RIBBON TABLEAUX AND THE HEISENBERG ALGEBRA

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Abstract. In [LLT] Lascoux, Leclerc and Thibon introduced symmetric functions $\mathcal{G}_\lambda$ which are spin and weight generating functions for ribbon tableaux. This article is aimed at studying these functions in analogy with Schur functions. In particular we will describe:

- a Pieri and dual-Pieri formula for ribbon functions,
- a ribbon Murnagham-Nakayama formula,
- ribbon Cauchy and dual Cauchy identities,
- and a $\mathbb{C}$-algebra isomorphism $\omega_n : \Lambda(q) \to \Lambda(q)$ which sends each $\mathcal{G}_\lambda$ to $\mathcal{G}_{\lambda'}$.

Our study of the functions $\mathcal{G}_\lambda$ will be connected to the Fock space representation $\mathbf{F}$ of $U_q(\widehat{\mathfrak{sl}}_n)$ via a linear map $\mathbf{\Phi} : \mathbf{F} \to \Lambda(q)$ which sends the standard basis of $\mathbf{F}$ to the ribbon functions. Kashiwara, Miwa and Stern [KMS] have shown that a copy of the Heisenberg algebra H acts on $\mathbf{F}$ commuting with the action of $U_q(\widehat{\mathfrak{sl}}_n)$. Identifying the Fock Space of H with the ring of symmetric functions $\Lambda(q)$ we will show that $\mathbf{\Phi}$ is in fact a map of H-modules with remarkable properties. We give a combinatorial proof that the ribbon Murnagham-Nakayama and Pieri rules are formally equivalent thus allowing us to describe the action of the genrators of H on $\mathbf{F}$ in terms of 'border ribbon strips'. We will also connect the ribbon Cauchy and Pieri formulae to the combinatorics of ribbon insertion as studied by Shimozono and White [SW2]. In particular we give complete combinatorial proofs for the domino $n = 2$ case.

Contents

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Introduction

Let $n \geq 1$ be a fixed integer and λ a partition with empty n-core. In analogy with the combinatorial definition of the Schur functions, Lascoux, Leclerc and Thibon [LLT] have defined a family of symmetric functions $\mathcal{G}_\lambda(X;q) \in \Lambda(q)$ by:

$$\mathcal{G}_\lambda(X;q) = \sum_T q^{s(T)} \mathbf{x}^{w(T)}$$

where the sum is over all *semistandard ribbon tableaux* of shape λ, and $s(T)$ and $w(T)$ are the spin and weight of T respectively. The definition of a semistandard ribbon tableaux is analagous to the definition of semistandard Young tableaux, with boxes replaced by ribbons (or border strips) of length n. We shall loosely call the functions $\mathcal{G}_\lambda(X;q)$ *ribbon functions*.

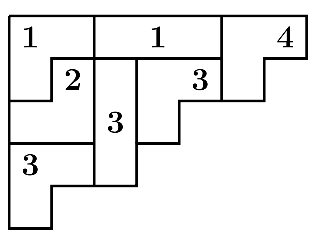

FIGURE 1. A semistandard 3-ribbon tableaux with shape $(7,6,4,3,1)$, weight $(2,1,3,1)$ and spin 7.

When $q = 1$ the ribbon functions become products of usual Schur functions. However, when the parameter q is introduced, it is no longer obvious that the functions $\mathcal{G}_\lambda(X;q)$ are symmetric. The main aim of this paper will be to develop the theory of ribbon functions in the same way Schur functions are studied in the ring of symmetric functions. In particular, we give:

- A ribbon Pieri formula (Theorem 29):
$$h_k[\left(1+q^2+\cdots+q^{2(n-1)}\right)X]\mathcal{G}_\nu(X;q)=\sum_\mu q^{s(\mu/\nu)}\mathcal{G}_\mu(X;q).$$
where the sum is over all μ such that μ/ν is a horizontal ribbon strip of size k. The notation $h_k[\left(1+q^2+\cdots+q^{2(n-1)}\right)X]$ denotes a plethysm.
- A ribbon Murnagham-Nakayam-rule (Theorem 23):
$$\left(1+q^{2k}+\cdots+q^{2k(n-1)}\right)p_k\mathcal{G}_\nu(X;q)=\sum_\mu \mathcal{X}_k^{\mu/\nu}(q)\mathcal{G}_\mu(X;q).$$
where $\mathcal{X}_k^{\mu/\nu}(q)$ is combinatorially defined as an alternating sum of spins over certain 'border n-ribbon strips' of size k.
- A Cauchy (and dual Cauchy) identity (Theorem 35):
$$\sum_\lambda \mathcal{G}_\lambda(X;q)\mathcal{G}_\lambda(Y;q)=\prod_{i,j}\prod_{k=0}^{n-1}\frac{1}{1-x_iy_jq^{2k}}$$
where the sum is over all partitions λ with a fixed n-core. This formula had been shown earlier [Lam] for the case $n=2$.
- A $\mathbb{C}$-algebra isomorphism $\omega_n:\Lambda(q)\to\Lambda(q)$ (Theorem 33) satisfying
$$\omega_n(\mathcal{G}_\lambda(X;q))=\mathcal{G}_{\lambda'}(X;q).$$
Even the existence of a linear map with such a property is not obvious as the functions $\mathcal{G}_\lambda$ are not linearly independent.

It is well known that the corresponding formulae are important for Schur functions in representation theory and algebraic geometry.

Much of the interest in the ribbon functions has been focused on the q-Littlewood Richardson coefficients $c_\lambda^\mu(q)$ of the expansion of $\mathcal{G}_\lambda(X;q)$ in the Schur basis:
$$\mathcal{G}_\lambda(X;q)=\sum_\mu c_\lambda^\mu(q)s_\lambda(X).$$
These are q-analogues of Littlewood Richardson coefficients. Using results of Varagnolo and Vasserot [VV], Leclerc and Thibon [LT] have shown that these coefficients are parabolic Kazhdan-Lusztig polynomials of type A. Results of Kashiwara and Tanisaki [KT] then imply that they are polynomials in q with non-negative coefficients. Much interest has also developed in connecting ribbon tableaux and the q-Littlewood Richardson coefficients to rigged configurations and the generalised Kostka polynomials defined by Kirillov and Shimozono [KS], Shimozono and Weyman [SW3], Schilling and Warnaar [SchW] and Shimozono [Shi]. We will focus mainly on the functions $\mathcal{G}_\lambda(X;q)$ though clearly our results imply relations between the $c_\lambda^\mu(q)$. One novelty is that we will study these functions even when the n-core is non empty and more generally the $\mathcal{G}_{\lambda/\mu}(X;q)$ for skew shapes λ/μ.

To prove that the functions $\mathcal{G}_\lambda(X;q)$ were symmetric Lascoux, Leclerc and Thibon connected them to Fock space representation $\mathbf{F}$ of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$. The crucial property of $\mathbf{F}$ is an action of a Heisenberg algebra H, commuting with the

action of $U_q(\widehat{\mathfrak{sl}}_n)$, discovered by Kashiwara, Miwa and Stern [KMS]. In particular, they showed that as a $U_q(\widehat{\mathfrak{sl}}_n) \times H$-module, $\mathbf{F}$ decomposes as

$$\mathbf{F} \cong V_{\Lambda_0} \otimes \mathbb{C}(q)[H_-]$$

where V_{Λ_0} is the highest weight representation of $U_q(\widehat{\mathfrak{sl}}_n)$ with highest weight Λ_0 and $\mathbb{C}(q)[H_-]$ is the usual Fock space representation of the Heisenberg algebra. We will give a description of the action of the genrators B_k of the H on $\mathbf{F}$ in terms of 'border ribbon strips' by giving a combinatorial proof that the ribbon Murnagham-Nakayama rule and the ribbon Pieri rules are formally equivalent (Theorem 54).

The connection between ribbon functions and the action of the Heisenberg algebra is made explicit by showing (Theorem 26) that the map $\mathbf{\Phi} : \mathbf{F} \to \mathbb{C}(q)[H_-]$ defined by

$$|\lambda\rangle \mapsto \mathcal{G}_\lambda$$

is a map of H-modules, after identifying $\mathbb{C}(q)[H_-]$ with the ring of symmetric functions $\Lambda(q)$ in the usual way. The map $\mathbf{\Phi}$ has the further remarkable property that it changes certain linear maps into algebra maps, as follows.

Lascoux, Leclerc and Thibon [LLT1] have constructed a global basis of $\mathbf{F}$ which extends Kashiwara's global crystal basis of V_{Λ_0}. They defined a bar involution on $\mathbf{F}$ which extends Kashiwara's involution on V_{Λ_0}. Another semi-linear involution, denoted $v \mapsto v'$ was also introduced and further studied in [LT] which satisfied the property

$$\langle \overline{u}, v \rangle = \langle u', \overline{v'} \rangle$$

for $u, v \in \mathbf{F}$ and $\langle |\lambda\rangle, |\mu\rangle \rangle = \delta_{\lambda\mu}$ the standard inner product on $\mathbf{F}$. We shall see that both involutions become algebra isomorphisms under the map $\mathbf{\Phi}$. In particular the 'image' of the involution $v \mapsto v'$ is simply ω_n (though it is not immediately clear that such an image can be well-defined) . We will also describe the image of certain vectors in the global basis under the map $\mathbf{\Phi}$.

Finally, we shall connect our study of ribbon functions to more combinatorial aspects of ribbon tableaux. For the easiest non trivial case of dominoes ($n = 2$), we shall prove all our main results using the domino insertion algorithms first studied by Barbasch, Vogan and Garfinkle [BV, Gar]. Shimozono and White [SW] subsequently generalised domino insertion to the semistandard case and also observed that the algorithm was compatible with the spin statistic on domino tableaux. In [Lam], the author described a dual-domino Schensted algorithm and observed both the Cauchy and dual-Cauchy identities. Here, we will prove the Pieri and Murnagham-Nakayama formulae using domino insertion.

Shimozono and White [SW2] have defined a ribbon-Schensted algorithm for $n > 2$ which is also compatible with spin on ribbon tableaux. As we shall discuss, this algorithm gives a combinatorial proof of the first ribbon Pieri formula for $k = 1$, but appears to be insufficient to prove either the Cauchy identity or the higher Pieri rules.

The combinatorial approach to ribbon tableaux has been relegated to a secondary role in our presentation. However, it should be noted that the investigation of the Heisenberg algebra was inspired by empirical calculations with domino and ribbon

tableaux made while writing [Lam]. Throughout the paper we will use classical symmetric function notation, however, most of our results could easily have been phrased in terms of the Heisenberg algebra H.

Organisation. The article begins with two introductory sections which give the notation we will use for tableaux and symmetric functions. The main body of the paper is split into three parts, which can be roughly described as being representation theory, symmetric function theory and combinatorics. The reader interested mostly in the representation theory will find that the first two parts can be read with nearly no references to the last part. In Part 1, we begin by describing the action of $U_q(\widehat{\mathfrak{sl}}_n)$ on the Fock space $\mathbf{F}$. The details of this action will rarely be used in the paper, but we present them for completeness. In Section 1.2 we will define the Heisenberg algebra and describe its action on both its usual Fock space representation $\mathbb{C}(q)[H_-]$ and on $\mathbf{F}$, as studied in [KMS, LLT, LT]. In Section 1.3 we will define the global basis of $\mathbf{F}$ and the two involutions introduced by Lascoux, Leclerc and Thibon. While this section is important for the overall understanding of the subject, it is logically independent of most of the proofs in Part 2 which mostly rely on the action of the Heisenberg algebra. Only one new result is present in Part 1, a description of the action of the generators B_k of the Heisenberg algebra on $\mathbf{F}$ in terms of border ribbon strips. In Part 2, we begin by describing the initial properties of ribbon functions. In Section 2.2 we will relate the global basis of $\mathbf{F}$ to the ribbon functions. In Section 2.3, we will prove the ribbon Murnagham-Nakayama rule. In Section 2.4 we define and study the map $\mathbf{\Phi} : \mathbf{F} \to \Lambda(q)$. In Section 2.5, the Pieri rule is shown modulo Theorem 54 of Section 3.2. In Section 2.6, we introduce the ribbon involution ω_n and prove its main properties. In Section 2.7, we prove the Cauchy and dual-Cauchy identities. In Section 2.8, we describe a ribbon inner product and study its relationship with another involution on $\Lambda(q)$. In Section 2.9, we prove a 'skew Cauchy identity', and also define super ribbon tableaux and functions. In Section 2.10, we discuss some open problems. Part 3 contains two rather separate sections. Section 3.1 discusses the relationship between the ribbon function formulae of Part 2 and ribbon insertion algorithms. Section 3.2 contains a standalone and purely combinatorial proof that the Murnagham-Nakyama and Pieri rules are formally equivalent.

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Partitions and Tableaux

In this section we give the notation and definitions we use for partitions and ribbon tableaux. A distinguished integer $n \geq 1$ will be fixed throughout the whole. When $n = 1$, the reader may check that we recover the classical theory of Schur functions.

A partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0)$ is a list of non-increasing integers. We will call l the length of λ, and denote it by $l(\lambda)$. We will say that λ is a partition of $\lambda_1 + \lambda_2 + \ldots + \lambda_l = |\lambda|$ and write $\lambda \vdash |\lambda|$. We will typically use λ, μ, ν and ρ to denote partitions. A composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ is an ordered list of non-negative

integers. As above, we will say that α is a composition of $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_l$. Let λ and μ be partitions. We will generally not distinguish between a partition λ and its corresponding Young diagram $D(\lambda)$. We will thus write $\lambda \subset \mu$ if $D(\lambda) \subset D(\mu)$. The skew shape λ/μ is the set difference between the corresponding diagrams of λ and μ when $\mu \subset \lambda$. The conjugate of a partition λ obtained by changing rows to columns, is denoted λ'. We will use the notation $m_k(\lambda)$ to denote the number of parts of λ equal to k.

A skew shape λ/μ is a horizontal strip if it contains at most one square in each column.

A skew shape λ/μ is a border strip if it is connected, and does not contain any 2×2 square. The height $h(b)$ of a border strip b is the number of rows in it, minus 1. A border strip tableaux is a chain of partitions

$$\mu^0 \subset \mu^1 \subset \cdots \subset \mu^r$$

such that each μ^{i+1}/μ^i is a border strip. The height of a border strip tableaux T is the sum of the heights of its border strips.

When a border strip has n squares for the distinguished (fixed) integer n, we will call it a ribbon. The height of the ribbon r will then be called its spin $s(r)$. The reader should be cautioned that in the literature the spin is usually defined as half of this.

A semistandard tableaux of shape λ/μ is a filling of each square (i, j) of the diagram $D(\lambda/\mu)$ with a positive integer such that the rows are non-decreasing and the columns are increasing. The weight $w(T)$ of such a tableaux T is the composition α such that α_i is the number of occurrences of i in T. The tableaux is standard if the numbers which occur are exactly those of $[m]$ for some integer m.

Let λ be a partition. Its n-core, obtained from λ by removal of n-ribbons (until we are no longer able to), is denoted $\tilde{\lambda}$. The n-quotient (see [Mac]) of λ will be denoted $(\lambda^{(0)}, \ldots, \lambda^{(n-1)})$. We shall write $\mathcal{P}$ for the set of partitions. We will use $\mathcal{P}_\delta$ to denote the set of partitions λ such that $\tilde{\lambda} = \delta$ for an n-core $\delta = \tilde{\delta}$.

A ribbon tableaux T of shape λ/μ is a tiling of λ/μ by n-ribbons and a filling of each ribbon with a positive integer (see Figure 1). If these numbers are exactly those of $[m]$, for some m, then the tableaux is called standard. We will use the convention that a ribbon tableaux of shape λ where $\tilde{\lambda} \neq \emptyset$ is simply a ribbon tableaux of shape $\lambda/\tilde{\lambda}$. A ribbon tableaux is semistandard if for each i

(1) removing all ribbons labelled j for $j > i$ gives a valid skew shape $\lambda_{\leq i}/\mu$ and,
(2) the subtableaux containing only the ribbons labelled i form a *horizontal n-ribbon strip.*

A horizontal n-ribbon strip is a skew shape tiled by ribbons such that the topright-most square of every ribbon touches the northern edge of the shape (see Figure 2). If such a tiling exists, it is necessarily unique.

We will often think of a ribbon tableaux as a chain of partitions

$$\tilde{\lambda} = \mu^0 \subset \mu^1 \subset \cdots \subset \mu^r = \lambda$$

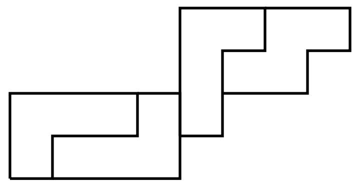

FIGURE 2. A horizontal 4-ribbon strip with spin 5.

where each μ^{i+1}/μ^i is a horizontal ribbon strip. The partitions μ^i here are not to be confused with the n-quotient of μ.

The spin $s(T)$ of a ribbon tableaux T is the sum of the spins of its ribbons. The cospin $\operatorname{cosp}(T)$ of a ribbon tableaux T is defined as $\operatorname{cosp}(T) = \operatorname{mspin}(sh(T)) - s(T)$, where $\operatorname{mspin}(sh(T))$ is the maximum spin of a ribbon tableaux of the same shape as T. The weight $w(T)$ of a tableaux is the composition counting the occurences of each value in T.

Littlewood's n-quotient map ([Lit], see also [SW1]) gives a weight preserving bijection between semistandard ribbon tableaux T of shape λ and n-tuples of semistandard Young tableaux $\left\{T^{(0)}, \ldots, T^{(n-1)}\right\}$ of shapes $\lambda^{(i)}$ respectively. Abusing language, we shall also refer to $\left\{T^{(0)}, \ldots, T^{(n-1)}\right\}$ as the n-quotient of T. Schilling, Shimozono and White [SSW] have described the cospin of a ribbon tableaux in terms of an inversion number of the n-quotient. None of our proofs will require the use of the n-quotient but occasionally we will comment on the $q = 1$ case for which the n-quotient will be important.

The n-quotient map can be described as follows. A diagonal $diag_d$ of a shape λ consists of all squares (i, j) such that $i - j = d$. If we draw all diagonals of the form $diag_{dn}$ then each ribbon will intersect each such diagonal exactly once. A ribbon's squares are linearly ordered from top right to bottom left. Suppose the diagonal $diag_{dn}$ intersects a ribbon r at the k^{th} square from the top right. Then the ribbon r is sent under the n-quotient map to a square in the diagonal $diag_d$ of $\lambda^{(k)}$. The numbers in the ribbon tableaux of Figure 1 have been placed along the diagonals $diag_{dn}$. Figure 3 shows its 3-quotient.

$$T^{(0)} = \begin{array}{|c|c|c|}\hline \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \hline \end{array} \qquad T^{(1)} = \begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{1} \\ \hline \mathbf{3} & \mathbf{3} \\ \hline \end{array} \qquad T^{(2)} = \emptyset$$

FIGURE 3. The 3-quotient of the ribbon tableaux T of Figure 1.

A horizontal ribbon strip can be described in terms of the n-quotient as a union of horizontal strips for each tableaux of the n-quotient.

Symmetric Functions

In this section we briefly review some standard notation in symmetric function theory. The reader is referred to [Mac] for further details.

Let $\Lambda_\mathbb{Z}$ denote the ring of symmetric functions with coefficients in $\mathbb{Z}$. Recall that $\Lambda_\mathbb{Z}$ has a distinguished integral basis s_λ known as the Schur functions. Nearly all the results of this paper can be stated in $\Lambda_\mathbb{Z}[q]$, but some intermediate steps may require working in $\Lambda = \Lambda_\mathbb{C}$ so we will use that as our symmetric function ring from now on. We will write $\Lambda(q)$ for $\Lambda_\mathbb{C} \otimes_\mathbb{C} \mathbb{C}(q)$.

It is well known that the Schur functions s_λ are orthogonal with respect to a natural inner product $\langle \, , \, \rangle$ on Λ and are unique up to signed permutation. We will denote the homogeneous, elementary, monomial and power sum symmetric functions by h_λ, e_λ, m_λ and p_λ respectively. Recall that we have $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$ and $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$ where $z_\lambda = 1^{m_1(\lambda)} m_1(\lambda)! 2^{m_2(\lambda)} m_2(\lambda)! \cdots$. Each of $\{p_i\}$, $\{e_i\}$ and $\{h_i\}$ generate Λ. We will write X to mean $(x_1, x_2, \ldots)$. Thus $s_\lambda(X) = s_\lambda(x_1, x_2, \ldots)$.

Recall that the Kostka matrix $K_{\lambda\mu}$ is defined as

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu.$$

We will denote the inverse Kostka matrix by $\kappa_{\lambda\mu}$:

$$m_\mu = \sum_\lambda \kappa_{\lambda\mu} s_\lambda.$$

Let $f \in \Lambda$. We will recall the definition of the plethysm $g \mapsto g[f]$. Write $g = \sum_\lambda c_\lambda p_\lambda$. Then we have

$$g[f] = \sum_\lambda c_\lambda \prod_{i=1}^{l(\lambda)} f(x_1^{\lambda_i}, x_2^{\lambda_i}, \ldots).$$

Thus the plethysm by f is the (unique) algebra isomorphism of Λ which sends $p_k \mapsto f(x_1^k, x_2^k, \ldots)$. When $f(x_1, x_2, \ldots; q) \in \Lambda(q)$ for a distinguished element q, we define the plethysm as $p_k \mapsto f(x_1^k, x_2^k, \ldots; q^k)$.

For example, the plethysm by $(1+q)p_1$ is given by sending

$$p_k \mapsto (1 + q^k) p_k$$

and extending to an algebra isomorphism $\Lambda(q) \to \Lambda(q)$. In such situations we will write $f[(1+q)X]$ for $f[(1+q)p_1]$.

We will be particularly concerned with the plethysm given by $(1+q^2+\cdots+q^{2n-2})p_1$. We will use $\Upsilon_{q,n}$ to denote the map $\Lambda(q) \to \Lambda(q)$ given by $f \mapsto f[(1 + q^2 + \cdots + q^{2n-2})X]$.

Part 1. The Fock Space of $U_q(\widehat{\mathfrak{sl}}_n)$ and the Heisenberg Algebra

1.1. The Fock Space representation $\mathbf{F}$ of $U_q(\widehat{\mathfrak{sl}}_n)$

In this section we introduce the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ and its (q-deformed) Fock Space representation $\mathbf{F}$. We will only be using the action of $U_q(\widehat{\mathfrak{sl}}_n)$ to define the canonical basis, but we include the details for completeness. A concise introduction to the material of this section can be found in [Lec]. Throughout q can be thought

of as either a formal parameter of as a generic complex number (not equal to a root of unity).

We denote by $\mathfrak{h}$ the 'Cartan' subalgebra of $\hat{\mathfrak{sl}}_n$ spanned over $\mathbb{C}$ by the basis $\{h_0, h_1, \ldots, h_{n-1}, D\}$. The dual basis will be spanned by $\{\Lambda_0, \Lambda_1, \ldots, \Lambda_{n-1}, \delta\}$. We set $\alpha_i = 2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1}$ for $i \in \{1, 2, \ldots, n-2\}$, and $\alpha_0 = \Lambda_0 - \Lambda_{n-1} - \Lambda_1 + \delta$ and $\alpha_{n-1} = -\Lambda_{n-2} + \Lambda_{n-1} - \Lambda_0$. The generalised Cartan matrix $[\langle \alpha_i, h_j \rangle]$ will be denoted a_{ij}. Set $P^\vee = (\oplus_{i=0}^{n-1} \mathbb{Z} h_i) \oplus \mathbb{Z} D$.

The algebra $U_q(\widehat{\mathfrak{sl}}_n)$ is the associative algebra over $\mathbb{C}(q)$ generated by elements e_i, f_i for $0 \le i \le n-1$, and q^h for $h \in P^\vee$ satisfying the following relations:

$$\begin{gathered}
q^h q^{h'} = q^{h+h'}, \\
q^{h_i} e_j q^{-h_i} = q^{a_{ij}} e_j, \quad q^{h_i} f_j q^{-h_i} = q^{-a_{ij}} f_j, \\
q^D e_i q^{-D} = \delta_{i0} q^{-1} e_0, \quad q^D f_i q^{-D} = \delta_{i0} q f_0, \\
[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \\
\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix} e_i^{1-a_{ij}-k} e_j e_i^k = 0 \quad (i \neq j), \\
\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix} f_i^{1-a_{ij}-k} f_j f_i^k = 0 \quad (i \neq j).
\end{gathered}$$

We have used the standard notation

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}}, \quad [k]! = [k][k-1] \cdots [1],$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[n-k]![k]!}.$$

The Fock Space $\mathbf{F}$ is an infinite dimensional vector space over $\mathbb{C}(q)$ spanned by a countable basis $|\lambda\rangle$ indexed by $\lambda \in \mathcal{P}$. We follow the terminology of [LLT].

There is an action of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{F}$ due to Hayashi [Hay] which was formulated essentially as follows by Misra and Miwa [MM].

Recall that a cell (i, j) has content given by $c(i, j) = i - j$. Its residue $p(i, j) \in \{0, 1, \ldots, n-1\}$ is then $i - j \bmod n$. We call (i, j) an indent k-node of λ if $p(i, j) = k$ and $\lambda \cup (i, j)$ is a valid Young diagram. We make the analogous definition for a removable i-node.

Let $i \in \{0, 1, \ldots, n-1\}$ and $\mu = \lambda \cup \delta$ for an indent i-node δ of λ. Now set

$N_i(\lambda) = \#\{$ indent i-nodes of $\lambda\} - \#\{$ removable i-nodes of $\lambda\}$.

$N_i^l(\lambda, \mu) = \#\{$ indent i-nodes of λ to the left of δ (not counting δ) $\} - \#\{$ removable i-nodes of λ to the left of $\delta\}$.

$N_i^r(\lambda, \mu) = \#\{$ indent i-nodes of λ to the right of δ (not counting δ) $\} - \#\{$ removable i-nodes of λ to the left of $\delta\}$.

$N^0(\lambda) = \#\{$ 0 nodes of $\lambda\}$.

Then we have the following theorem.

Theorem 1. *The following formulae define an action of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ on* $\mathbf{F}$*:*

$$\begin{aligned} q^{h_i}|\lambda\rangle &= q^{N_i(\lambda)}|\lambda\rangle, \text{ for each } i \in \{0,1,\ldots,n-1\}, \\ q^D|\lambda\rangle &= q^{N^0(\lambda)}|\lambda\rangle, \\ f_i|\lambda\rangle &= \sum_\mu q^{N_i^r(\lambda,\mu)}|\mu\rangle, \text{ summed over all } \mu \text{ such that } \mu/\lambda \text{ has residue } i, \\ e_i|\lambda\rangle &= \sum_\mu q^{-N_i^l(\lambda,\mu)}|\mu\rangle, \text{ summed over all } \mu \text{ such that } \lambda/\mu \text{ has residue } i. \end{aligned}$$

The $U_q(\widehat{\mathfrak{sl}}_n)$-submodule of $\mathbf{F}$ generated by the vector $|0\rangle$ is easily seen to be the irreducible highest weight module with highest weight Λ_0, which we will denote V_{Λ_0}.

1.2. The Action of the Heisenberg Algebra

This action of the Heisenberg Algebra on $\mathbf{F}$ will be essential for our study of ribbon functions in Part 2.

The Heisenberg Algebra H will be the associative algebra with 1 generated over $\mathbb{C}(q)$ by a countable set of generators $B_k : k \in \mathbb{Z} - \{0\}$ satisfying

$$[B_k, B_l] = l a_l(q)\delta_{k,-l} \tag{1}$$

for some elements $a_l(q) \in \mathbb{C}(q)$ satisfying $a_l(q) = a_{-l}(q)$. (Often the element 1 is called the central element and denoted c, but we will not need this generality). The Fock Space representation $\mathbb{C}(q)[H_-]$ of H is the polynomial algebra

$$\mathbb{C}(q)[H_-] \cong \mathbb{C}(q)[B_{-1}, B_{-2}, \ldots].$$

The elements B_{-k} for $k \geq 1$ act by multiplication on $\mathbb{C}(q)[H_-]$. The action of B_k for $k \geq 1$ is given by (1) and the relation

$$B_k \cdot 1 = 0 \text{ for } k \geq 1.$$

One common explicit construction of $\mathbb{C}(q)[H_-]$ is given by $\Lambda(q)$. We may identify B_k as the following operators:

$$B_{-k} : f \mapsto a_k(q)p_k \cdot f \text{ for } k \geq 1$$

and

$$B_k : f \mapsto k\frac{\partial}{\partial p_k} f \text{ for } k \geq 1.$$

Under this identification, the operators B_k have degree $-k$. The reason for splitting $a_k(q)$ and k is because p_k and $k\frac{\partial}{\partial p_k}$ are adjoint operators under the usual inner product $\langle\, , \rangle$ on $\Lambda(q)$ (see for example [Mac]).

A standard lemma that we shall need later is

Lemma 2. *Let $k \geq 1$ be an integer and λ be a partition. Then*

$$B_k B_{-\lambda} = k a_k(q) m_k(\lambda) B_{-\mu} + B B_k$$

where $m_k(\lambda)$ is the number of parts of λ equal to k and B is some element in H and μ is λ with one less part equal to k. If $m_k(\lambda) = 0$ to begin with then the first term is just 0.

Proof. We may commute B_k with $B_{-\lambda_i}$ immediately for parts $\lambda_i \neq k$. For each part equal to k, using the relation $[B_{-k}, B_k] = k a_k(q)$ introduces one term of the form $k a_k(q) B_{-\mu}$. □

Kashiwara, Miwa and Stern [KMS] have defined an action of the the affine Hecke algebra $\hat{H}_N$ on the tensor product $V(z)^{\otimes N}$ of evaluation modules. As $N \to \infty$, we obtain an action of the center $Z(\hat{H}_N)$ as a copy of the Heisenberg algebra H on $\mathbf{F}$, commuting with the action of $U_q(\widehat{\mathfrak{sl}}_n)$. The operators B_k are given as the infinite power sums

$$B_k = \sum_{i=1}^{\infty} Y_i^{-k}$$

in terms of the certain generators Y_i (in the $\{T_i, Y_j\}$ presentation) of $\hat{H}_N$.

The following theorems are due to Kashiwara, Miwa and Stern [KMS].

Theorem 3. *The operators B_k commute with the action of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$. They satisfy the relations*

$$[B_k, B_l] = k \frac{1 - q^{-2nk}}{1 - q^{-2k}} \delta_{k,-l}$$

and generate a copy of the Heisenberg algebra.

We shall see later in Section 3.1 that the factor $\left(\frac{1-q^{-2nk}}{1-q^{-2k}}\right)$ can be given a combinatorial explanation in terms of ribbon insertion.

Theorem 4. *The Fock space $\mathbf{F}$, regarded as a representation of $U_q(\widehat{\mathfrak{sl}}_n) \otimes U(H)$ decomposes as the tensor product*

$$\mathbf{F} \simeq V_{\Lambda_0} \otimes \mathbb{C}(q)[H_-]$$

where $\mathbb{C}(q)[H_-]$ is the Fock space of the Heisenberg algebra H and V_{Λ_0} is the highest weight representation with highest weight Λ_0.

We write $B_\alpha = B_{\alpha_l} B_{\alpha_{l-1}} \cdots B_{\alpha_1}$ for a composition α. Similarly, $B_{-\alpha}$ denotes $B_{-\alpha_l} \cdots B_{-\alpha_1}$.

Lascoux, Leclerc and Thibon [LLT] have described the action of certain elements $\mathcal{U}_k$, $\tilde{\mathcal{U}}_k$, $\mathcal{V}_k$, and $\tilde{\mathcal{V}}_k$ of H on the Fock Space $\mathbf{F}$, in terms of ribbon tableaux.

In terms of the elements $Y_i^{\pm 1}$ of the affine Hecke algebra, we have

$$\begin{aligned}\mathcal{U}_k &= h_k(Y_1, Y_2, \ldots),\\ \mathcal{V}_k &= h_k(Y_1^{-1}, Y_2^{-1}, \ldots),\\ \tilde{\mathcal{U}}_k &= e_k(Y_1, Y_2, \ldots),\\ \tilde{\mathcal{V}}_k &= e_k(Y_1^{-1}, Y_2^{-1}, \ldots),\end{aligned}$$

where the e_k and h_k are the elementary and homogeneous symmetric functions. To avoid mention of the elements Y_i one may write them as

$$\mathcal{U}_k = \sum_\lambda b_{k,\lambda} B_\lambda$$

where the coefficients $b_{k,\lambda}$ are given by the expansion

$$h_k = \sum b_{k,\lambda} p_\lambda$$

in the ring Λ of symmetric functions and p_λ are the power sum symmetric functions.

Proposition 5. *The elements $\mathcal{U}_k$, $\tilde{\mathcal{U}}_k$, $\mathcal{V}_k$, and $\tilde{\mathcal{V}}_k$ of $\mathcal{H}$ act on $\mathbf{F}$ as linear operators defined by*

$$\mathcal{V}_k|\lambda\rangle = \sum_\mu (-q)^{-s(\mu/\lambda)}|\mu\rangle,$$

where the sum is over all μ such that μ/λ is a horizontal n-ribbon strip of size k. Similarly,

$$\mathcal{U}_k|\lambda\rangle = \sum_\nu (-q)^{-s(\lambda/\nu)}|\nu\rangle,$$

summed over all ν such that λ/ν is a horizontal n-ribbon strip of size k. The formulae for $\tilde{\mathcal{U}}_k$ and $\tilde{\mathcal{V}}_k$ are exactly analagous, with horizontal ribbon strips replaced by vertical ribbon strips.

We will write $\mathcal{V}_\alpha$ for $\mathcal{V}_{\alpha_l} \cdots \mathcal{V}_{\alpha_1}$ and similarly for $\mathcal{U}_\alpha$, $\tilde{\mathcal{V}}_\alpha$ and $\tilde{\mathcal{U}}_\alpha$. Thus

$$\mathcal{V}_\alpha|\mu\rangle = \sum_T q^{s(T)}|\lambda\rangle$$

summed over all semistandard ribbon tableaux of shape λ/μ and weight α.

In Part 3 we will show (Theorem 54) that the Pieri style rule of Proposition 5 formally implies a Murnagham-Nakayama style rule for the the operators B_k and B_{-k}. The precise statement and proof of Theorem 54 has been deferred until the end as its proof is completely combinatorial and unrelated to the Fock space.

Proposition 6. *The linear operators B_{-k} for $k \geq 1$ act on $\mathbf{F}$ as*

$$B_{-k}|\lambda\rangle = \sum_\mu \mathcal{X}_k^{\mu/\lambda}(-q^{-1})|\mu\rangle$$

where $\mathcal{X}_k^{\mu/\lambda}(q)$ *is given by*

$$\mathcal{X}_k^{\mu/\lambda}(q) = \sum_S (-1)^{h(S)} q^{s(S)}$$

summed over all border ribbon strip tilings S *of* μ/λ. *Similarly,*

$$B_k|\lambda\rangle = \sum_\mu \mathcal{X}_k^{\lambda/\mu}(-q^{-1})|\mu\rangle.$$

Proof. This is an immediate consequence of Proposition 5 and Theorem 54. □

Border ribbon strips will be defined combinatorially later (Definition 15) and are ribbon analogues of usual border strips.

It follows immediately from the above propositions and Theorem 3 that the sets $\{\mathcal{V}_\lambda|0\rangle\}_{\lambda\in\mathcal{P}}$, $\{\mathcal{U}_\lambda|0\rangle\}_{\lambda\in\mathcal{P}}$ and $\{B_\lambda|0\rangle\}_{\lambda\in\mathcal{P}}$ form bases of the space of highest weight vectors of $U_q(\widehat{\mathfrak{sl}}_n)$ in $\mathbf{F}$.

1.3. Global Bases of F

We first define a bar involution $v \mapsto \overline{v}$ on $\mathbf{F}$ following Leclerc and Thibon [LT, LT1]. This involution restricted to V_{Λ_0} (the $U_q(\widehat{\mathfrak{sl}}_n)$ submodule with highest weight vector $|0\rangle$) agrees with Kashiwara's involution [Kas].

Proposition 7. *There exists a unique semi-linear map* $^-: \mathbf{F} \to \mathbf{F}$ *satisfying*

$$\begin{aligned}
\overline{qv} &= q^{-1}\overline{v},\\
\overline{f_i \cdot v} &= f_i \cdot \overline{v},\\
\overline{e_i \cdot v} &= e_i \cdot \overline{v},\\
\overline{B_{-k} \cdot v} &= B_{-k} \cdot \overline{v},\\
\overline{B_k \cdot v} &= q^{2(n-1)k} B_k \cdot \overline{v}.
\end{aligned}$$

The Fock space $\mathbf{F}$ has a natural order '$<$' defined by $|\lambda\rangle < |\mu\rangle$ if and only if $\lambda \prec \mu$ in dominance order. Leclerc and Thibon show that $^-$ is triangular with respect to the basis $|\lambda\rangle$ and conclude that the global basis of the following theorem exists.

Theorem 8. *There exist unique vectors* $G_\lambda \in \mathbf{F}$ *for* $\lambda \in \mathcal{P}$ *satisfying:*

$$\overline{G_\lambda} = G_\lambda$$

and

$$G_\lambda \equiv |\lambda\rangle \ mod\ q^{-1}\mathcal{L}^-$$

where $\mathcal{L}^-$ *is the* $\mathbb{Z}[q^{-1}]$ *submodule of* $\mathbf{F}$ *spanned by* $|\lambda\rangle$.

When we restrict this to the $U_q(\widehat{\mathfrak{sl}}_n)$ submodule V_{Λ_0} of $\mathbf{F}$ generated by $|0\rangle$, the G_λ is essentially the global upper crystal basis of V_{Λ_0} (see [LLT1, Kas1]). This follows from the fact that the bar involution agrees with Kashiwara's involution when restricted to V_{Λ_0}.

Some of the G_λ are especially easy to describe in terms of the action of the Heisenberg algebra. In particular, in analogy with Steinberg's tensor product theorem, Leclerc and Thibon [LT] show that $G_\mu = S_\lambda|\nu\rangle$ for an n-regular partition ν and $\mu = n\lambda + \nu$. We will only need the following special case.

Proposition 9. *Let $S_\lambda = \sum_\mu \chi^\lambda_\mu B_{-\lambda}$. Thus in terms of the elements Y_i we have $S_\lambda = s_\lambda(Y_1^{-1}, Y_2^{-1}, \ldots)$. Then we have*

$$G_{n\lambda} = S_\lambda|0\rangle.$$

Proof. It is clear that $\overline{|0\rangle} = |0\rangle$. Thus by Proposition 7 we have $\overline{S_\lambda|0\rangle} = S_\lambda|0\rangle$. By the definition of $G_{n\lambda}$ it suffices to show that $S_\lambda|0\rangle \equiv |n\lambda\rangle \mod q^{-1}\mathcal{L}^-$.

By Proposition 5 we know that

$$\mathcal{V}_\alpha|0\rangle \equiv \sum_T |sh(T)\rangle \mod q^{-1}\mathcal{L}^-$$

where the sum is over all ribbon tableaux of spin 0 and weight α. It is clear that these are in bijection with usual semistandard Young tableaux of weight α. Thus

$$\mathcal{V}_\alpha|0\rangle \equiv \sum_\lambda K_{\lambda\alpha}|n\lambda\rangle \mod q^{-1}\mathcal{L}^-.$$

Comparing with $\mathcal{V}_\alpha = \sum_\lambda K_{\lambda\alpha}S_\lambda$ we see that

$$S_\lambda|0\rangle \equiv |n\lambda\rangle \mod q^{-1}\mathcal{L}^-$$

which completes the proof. □

It follows immediately that $\{G_{n\lambda}\}$ form a basis of the space of highest weight vectors of the action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{F}$.

Remark 10. (1) Let

$$G_\lambda = \sum_\mu l_{\lambda,\mu}(-q^{-1})|\mu\rangle.$$

Varagnolo and Vasserot [VV] have shown that $l_{\lambda,\mu}$ is a parabolic Kazhdan-Lusztig polynomial for the affine Hecke algebra of type A. These polynomials were introduced by Deodhar [D1, D2] and shown to have non-negative coefficients by Kashiwara and Tanisaki [KT].

(2) We have not mentioned the lower global basis G^+ of $\mathbf{F}$ as they are less related to the ribbon functions we will be studying. The G^+_λ are defined in a similar way to the G_λ by

$$G^+_\lambda = \overline{G^+_\lambda}$$
$$G^+_\lambda \equiv |\lambda\rangle \bmod q\mathcal{L}^+$$

where $\mathcal{L}^+$ is the $\mathbb{Z}[q]$-submodule spanned by $|\lambda\rangle$.

Finally, we will be needing a semi-linear involution $v \mapsto v'$ on $\mathbf{F}$. This is defined by $q' = q^{-1}$ and

$$|\lambda\rangle \mapsto |\lambda'\rangle.$$

Then we have [LT, Proposition 7.10]

Proposition 11. *For all $v \in \mathbf{F}$ and compositions α satisfying $|\alpha| = k$ we have*

$$(e_i u)' = q^{h_{-i}-1} e_{-i} u', \qquad (f_i u)' = q^{-h_{-i}-1} f_{-i} u',$$
$$(\mathcal{V}_\beta u)' = (-q)^{(n-1)k} \tilde{\mathcal{V}}_\beta u', \qquad (\mathcal{U}_\beta u)' = (-q)^{(n-1)k} \tilde{\mathcal{U}}_\beta u'.$$

One immediate consequence of this and Proposition 9 is that

$$(G_{n\lambda})' = (-q)^{(n-1)k} G_{n\lambda'}.$$

We shall see later an explicit connection between the global basis and ribbon tableaux. This should come as no surprise: the $\mathcal{V}_\alpha$ are described combinatorially in terms of ribbon tableaux and S_λ can be expressed in terms of the $\mathcal{V}_\alpha$ (via the inverse Kostka matrix).

Part 2. Ribbon Functions

2.1. Definitions and Initial Properties

We will now define the central objects of this paper as introduced by Lascoux, Leclerc and Thibon in [LLT].

Definition 12. Let λ/μ be a skew partition, tileable by n-ribbons. Define the symmetric functions $\mathcal{G}_{\lambda/\mu} \in \Lambda(q)$ as:

$$\mathcal{G}_{\lambda/\mu}(X; q) = \sum_T q^{s(T)} \mathbf{x}^{w(T)}$$

where the sum is over all semistandard ribbon tableaux T of shape λ/μ and $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. When λ is a partition with non-empty n-core, we write $\mathcal{G}_\lambda$ for $\mathcal{G}_{\lambda/\tilde{\lambda}}$. These functions will be loosely called *ribbon functions*.

The fact that the functions $\mathcal{G}_{\lambda/\mu}$ are symmetric is not obvious from the combinatorial definition. However, using the action of the Heisenberg algebra on the Fock space $\mathbf{F}$, the proof is immediate ([LLT]) and reproduced below.

Definition 13. Let λ/μ be a skew shape tileable by n-ribbons. Then define

$$\mathcal{K}_{\lambda/\mu,\alpha}(q) = \sum_T q^{s(T)},$$

the spin generating function of all semistandard ribbon tableaux T of shape λ/μ and weight α. Similarly let

$$\mathcal{L}_{\lambda/\mu,\alpha}(q) = \sum_T q^{s(T)}$$

summed over all column semistandard ribbon tableaux of shape λ/μ and weight α. A ribbon tableaux is column semistandard if its conjugate is semistandard.

Thus

$$\mathcal{G}_{\lambda/\mu}(X; q) = \sum_\alpha \mathcal{K}_{\lambda/\mu,\alpha}(q) \mathbf{x}^\alpha.$$

Theorem 14. *The functions $\mathcal{G}_{\lambda/\mu}(X;q)$ are symmetric functions.*

Proof. A semistandard ribbon tableaux can be expressed as a chain of partitions differing by horizontal ribbon strips. Thus

$$\mathcal{V}_\alpha|\mu\rangle = \sum_\nu \mathcal{K}_{\nu/\mu,\alpha}(-q^{-1})|\nu\rangle.$$

But if β is a permutation of α, then $\mathcal{V}_\alpha = \mathcal{V}_\beta$ since the $\mathcal{V}_k$ commute. This shows that

$$\mathcal{K}_{\lambda/\mu,\alpha}(q) = \mathcal{K}_{\lambda/\mu,\beta}(q),$$

after equating coefficients of λ. □

We will also need the following definition of a *border strip ribbon tableaux*.

Definition 15. A *border ribbon strip* T is a connected skew shape λ/μ with a distinguished tiling by disjoint non-empty horizontal ribbon strips $T_1, \ldots, T_a$ such that the diagram $T_{+i} = \cup_{i \leq j} T_i$ is a valid skew shape for every i and for each connected component C of T_i we have

(1) The shape of $C \cup T_{i-1}$ is not a horizontal ribbon strip. Thus C has to 'touch' T_{i-1} 'from below'.
(2) No sub horizontal ribbon strip C' of C which can be added to T_{i-1} satisfies the above property. Since C is connected, this is equivalent to saying that only the rightmost ribbon of C touches T_{i-1}.

We further require that T_1 is connected. The height $h(T_i)$ of the horizontal ribbon strip T_i is the number of its components. The height $h(T)$ of the border ribbon strip is defined as $h(T) = (\sum_i h(T_i)) - 1$. The size of the border ribbon strip T is then the total number of ribbons in $\cup_i T_i$. A border ribbon strip tableaux is a chain $T = \lambda_0 \subset \lambda_1 \cdots \subset \lambda_r$ of shapes such that λ_i/λ_{i-1} has been given the structure of a border ribbon strip. The type of $T = \{\lambda_i\}$ is then the composition α with α_i equal to the size of λ_i/λ_{i-1}.

Define $\mathcal{X}_\nu^{\mu/\lambda}$ as

$$\mathcal{X}_\nu^{\mu/\lambda}(q) = \sum_T (-1)^{h(T)} q^{s(T)}$$

summed over all border ribbon strip tableaux of shape μ/λ and type ν.

Note that this definition reduces to the usual definition of a border strip and border strip tableaux when $n = 1$, in which case all the horizontal strips T_i are actually connected.

Example 16. Let $n = 2$ and $\lambda = (4,2,2,1)$. Suppose S is a border ribbon strip such that S_1 has shape $(7,5,2,1)/(4,2,2,1)$, and thus it has size 3 and spin 1. We will now determine all the possible horizontal ribbon strips which may form S_2. It suffices to find the possible connected components that may be added. The domino $(9,5,2,1)/(7,5,2,1)$ may not be added since its union with S_1 is a horizontal ribbon strip, violating the conditions of the definition. The domino strip $(8,8,2,1)/(7,5,2,1)$ is not allowed since the domino $(8,8,2,1)/(7,7,2,1)$ can be removed and we still obtain a strip which touches S_1.

The legitimate connected horizontal ribbon strips C which can be added are $(7,7,2,1)/(7,5,2,1)$, $(7,5,4,1)/(7,5,2,1)$ and $(7,5,3,3,2,1)/(7,5,2,1)$ as shown in Figure 4. Thus assuming S_2 is non-empty, there are 5 choices for S_2, corresponding to taking some compatible combination of the three connected horizontal ribbon strips above.

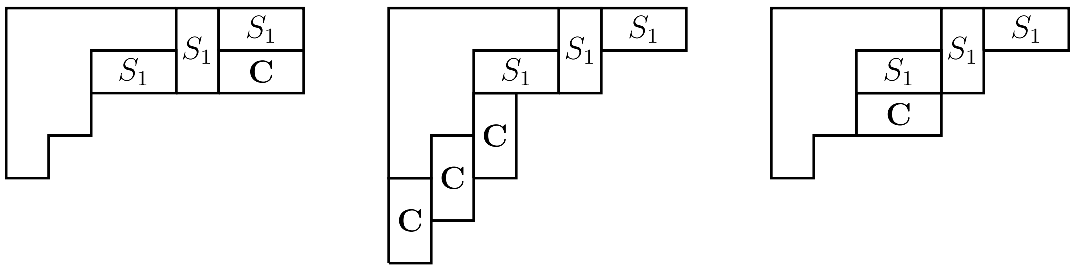

FIGURE 4. Connected horizontal strips C which can be added to $S_1 = (7,5,2,1)/(4,2,2,1)$ to form a border ribbon strip. The resulting border ribbon strips all have height 1.

Example 17. As before let $n = 2$. We will calculate $\mathcal{X}_5^{\lambda/\mu}(q)$ for $\lambda = (5,5,2)$ and $\mu = (2)$. The relevant border ribbon strips S are (successive differences of the following chains denote the S_i)

- $(2) \subset (5,5,2)$ with height 0 and spin 5,
- $(2) \subset (5,3,2) \subset (5,5,2)$ with height 1 and spin 3,
- $(2) \subset (5,5) \subset (5,5,2)$ with height 1 and spin 3,
- $(2) \subset (5,3) \subset (5,5,2)$ with height 2 and spin 1.

Thus

$$\mathcal{X}_5^{\lambda/\mu}(q) = q^5 - 2q^3 + q.$$

When $q = 1$, the ribbon functions become products of Schur functions:

$$\mathcal{G}_\lambda(X;1) = s_{\lambda^{(0)}} s_{\lambda^{(1)}} \cdots s_{\lambda^{(n-1)}}.$$

This is a consequence of Littlewood's n-quotient map. In fact, up to sign, $\mathcal{G}_\lambda(X;1)$ is essentially $\phi_n(s_\lambda)$ where ϕ_n is the adjoint operator to taking the plethysm by p_n. More generally, $\mathcal{G}_{\lambda/\mu}$ reduces to a product of skew Schur functions at $q = 1$:

$$\mathcal{G}_{\lambda/\mu}(X;1) = s_{\lambda^{(0)}/\mu^{(0)}} s_{\lambda^{(1)}/\mu^{(1)}} \cdots s_{\lambda^{(n-1)}/\mu^{(n-1)}}.$$

Remark 18. Another set of symmetric functions $\mathcal{H}_\lambda(X;q)$ are defined ([LLT]) by

$$\mathcal{H}_\lambda(X;q) = \mathcal{G}_{n\lambda}(X;q).$$

It is not hard to see that

$$\mathcal{H}_\lambda(X;0) = s_\lambda$$

and that

$$\mathcal{H}_\lambda(X;1) = s_\lambda + \sum_{\mu \prec \lambda} s_\mu$$

where $\prec$ denotes the usual dominance order on partitions. Thus the functions $\mathcal{H}_\lambda(X;q)$ form a basis of $\Lambda(q)$ over $\mathbb{C}(q)$. In [LLT] it is shown that the cospin versions $\tilde{\mathcal{H}}_\lambda(X;q)$ generalise the modified Hall-Littlewood functions $Q'(X;q)$. We shall however be concerned mainly with the functions $\mathcal{G}_\lambda(X;q)$.

2.2. Global Bases and Ribbon Functions

In this section we reproduce (in slightly more generality) a calculation due to Leclerc and Thibon [LT] which relates the q-Littlewood Richardson coefficients to the global bases $G_{n\lambda}$ of $\mathbf{F}$. The q-Littlewood Richardson coefficients $c_\mu^\lambda(q)$ are defined by the expansion of the ribbon functions on the Schur basis.

Definition 19. Let λ/μ be a skew shape tileable by n-ribbons. Define the polynomials $c^\nu_{\lambda/\mu}(q)$ by

$$\mathcal{G}_{\lambda/\mu}(X;q) = \sum_\nu c^\nu_{\lambda/\mu}(q) s_\nu(X).$$

As usual, we abuse notation by writing $c^\nu_\lambda(q)$ for $c^\nu_{\lambda/\tilde{\lambda}}(q)$ for partitions λ with non-empty n-core.

When λ has empty n-core, the q-Littlewood Richardson coefficients are usually written in terms of the n-quotient:

$$c^\nu_\lambda(q) = c^\nu_{\lambda^{(0)},\ldots,\lambda^{(n-1)}}(q).$$

We can connect the q-Littlewood Richardson coefficients to the Heisenberg algebra immediately.

Lemma 20. *Let λ and μ be partitions. Then*

$$S_\lambda|\mu\rangle = \sum_\nu c^\lambda_{\nu/\mu}(-q^{-1})|\nu\rangle$$

where the sum is over all partitions ν such that ν/μ is tileable by n-ribbons.

Proof. We have by definition

$$\mathcal{G}_{\nu/\mu}(X;q) = \sum_\rho \mathcal{K}_{\nu/\mu,\rho}(q) m_\rho = \sum_\lambda \left(\sum_\rho \mathcal{K}_{\nu/\mu,\rho}(q)\kappa_{\lambda\rho}\right) s_\lambda$$

where $\kappa_{\lambda\rho}$ is given by

$$m_\rho = \sum_\lambda \kappa_{\lambda\rho} s_\lambda.$$

Thus

$$c^\lambda_{\nu/\mu}(q) = \sum_\rho \mathcal{K}_{\nu/\mu,\rho}(q)\kappa_{\lambda\rho}.$$

By standard results in symmetric function theory we also have $s_\lambda = \sum_\rho \kappa_{\lambda\rho} h_\rho$. Hence using Proposition 5,

$$\begin{aligned} S_\lambda|\mu\rangle &= \sum_\rho \kappa_{\lambda\rho}\mathcal{V}_\rho|\mu\rangle \\ &= \sum_\nu \left(\sum_\rho \kappa_{\lambda\rho}\mathcal{K}_{\nu/\mu,\rho}(-q^{-1})\right)|\nu\rangle \\ &= \sum_\nu c^\lambda_{\nu/\mu}(-q^{-1})|\nu\rangle. \end{aligned}$$

□

Corollary 21. *Let λ be a partition. Then*

$$G_{n\lambda} = \sum_\mu c^\lambda_\mu(-q^{-1})|\mu\rangle$$

summed over partitions μ with no n-core.

Proof. This follows from Lemma 20 and Proposition 9. □

By Remark 10, we also see that the polynomials $c^\lambda_\mu(q) = c^\lambda_{\mu^{(0)},\ldots,\mu^{(n-1)}}(q)$ have non-negative coefficients. For $n = 2$, there is a combinatorial interpretation for the coefficients in terms of Yamanouchi domino tableaux (see [CL]).

2.3. The Murnagham-Nakayama Rule

The core calculation of this paper will be the ribbon Murnagham-Nakayama Rule, which is essentially a consequence of the fact that the B_k act on $\mathbf{F}$ as a copy of the Heisenberg algebra. We will begin by reminding the reader of the classical Murnagham-Nakayama Rule.

Let

$$s_\lambda(X) = \sum_\mu z_\mu^{-1}\chi^\lambda_\mu p_\mu$$

be the expansion of the Schur functions in the power sum basis. When $\mu = (k)$ has only one part then we will write χ^λ_k for χ^λ_μ. The coefficients χ^λ_μ are the values of the character of $S_{|\lambda|}$ indexed by λ on the conjugacy class indexed by μ. The classical Murnagham-Nakayama rule gives a combinatorial interpretation of these numbers:

$$\chi^\lambda_\mu = \sum_T (-1)^{h(T)}$$

where the sum is over all border-strip tableaux of shape λ and type μ. The numbers χ^λ_μ are in fact the characters of the irreducible representation labelled by λ of the symmetric group $S_{|\lambda|}$, where μ is the type of the conjugacy class. Thus in particular all the irreducible characters of the symmetric groups take values in the integers. See for example [EC2, Ch 7.18].

More generally, we have (see [EC2, Mac])

Proposition 22. *Let λ be a partition and α be a composition. Expand*
$$p_\alpha s_\lambda(X) = \sum_\mu \chi_\alpha^{\mu/\lambda} s_\mu(X).$$
Then $\chi_\alpha^{\mu/\lambda}$ is given by
$$\chi_\alpha^{\mu/\lambda} = \sum_T (-1)^{h(T)}$$
where the sum is over all border strip tableaux T of shape μ/λ and type α. Note: the border strip tableaux here should not be confused with ribbon tableaux. A border strip tableaux may have border strips of different sizes. A ribbon tableaux has all ribbons of length n.

This result is usually shown algebraically using the expression of the Schur function as a bialternant $s_\lambda = a_{\lambda+\delta}/a_\delta$. Theorem 54 implies that it is in fact a formal combinatorial consequence of the Pieri formula.

We will now give the analogue of Proposition 22 for ribbon functions.

Theorem 23 (Ribbon Murnagham-Nakayama Rule)**.** *Let $k \geq 1$ be an integer and ν be a partition. Then*
$$\left(1+q^{2k}+\cdots+q^{2k(n-1)}\right) p_k \mathcal{G}_\nu(X;q) = \sum_\mu \mathcal{X}_k^{\mu/\nu}(q)\mathcal{G}_\mu(X;q). \tag{2}$$
Also
$$k\frac{\partial}{\partial p_k}\mathcal{G}_\nu(X;q) = \sum_\mu \mathcal{X}_k^{\nu/\mu}(q)\mathcal{G}_\mu(X;q).$$

Proof. Let $\delta = \tilde{\nu}$ be the n-core of ν, which we fix throughout. Note that the only terms μ which occur in (2) satisfy $\tilde{\mu} = \tilde{\nu}$. Recall that we will often be writing μ instead of μ/δ for convenience.

We will calculate the expression $B_k S_\lambda|\delta\rangle$ with $k \geq 1$ in two ways. By Lemma 20 we can write
$$B_k S_\lambda|\delta\rangle = \sum_{\mu\in\mathcal{P}_\delta} c_\mu^\lambda(-q^{-1}) B_k|\mu\rangle$$
and by Proposition 6 this can be written as
$$\sum_\mu c_\mu^\lambda(-q^{-1})\left(\sum_\nu \mathcal{X}_k^{\mu/\nu}(-q^{-1})|\nu\rangle\right) = \sum_\nu\left(\sum_\mu c_\mu^\lambda(-q^{-1})\mathcal{X}_k^{\mu/\nu}(-q^{-1})\right)|\nu\rangle.$$

On the other hand, we know by Theorem 3 that B_k and S_λ are both operators within a copy of the Heisenberg Algebra. Thus we can compute $B_k S_\lambda$ within H. Write
$$S_\lambda = \sum_\mu \chi_\mu^\lambda B_{-\mu}.$$
By Lemma 2 and the fact that $B_k|\delta\rangle = 0$ (Proposition 6) we have
$$B_k S_\lambda|\delta\rangle = \left(\frac{1-q^{-2nk}}{1-q^{-2k}}\right)\sum_\mu \chi_k^{\lambda/\mu} S_\mu|\delta\rangle$$

since $p_k^\perp s_\lambda = \sum_\mu \chi_k^{\lambda/\mu} s_\mu$ in Λ. By Lemma 20 again, we find that this is equal to

$$\left(\frac{1-q^{-2nk}}{1-q^{-2k}}\right)\sum_\mu \chi_k^{\lambda/\mu} \sum_{\nu\in\mathcal{P}_\delta} c_\nu^\mu(-q^{-1})|\nu\rangle = \sum_{\nu\in\mathcal{P}_\delta}\left(\left(\frac{1-q^{-2nk}}{1-q^{-2k}}\right)\sum_\mu \chi_k^{\lambda/\mu} c_\nu^\mu(-q^{-1})\right)|\nu\rangle.$$

Equating coefficients of $|\nu\rangle$ we obtain

$$\left(\frac{1-q^{-2nk}}{1-q^{-2k}}\right)\sum_\mu \chi_k^{\lambda/\mu} c_\nu^\mu(-q^{-1}) = \sum_\mu c_\mu^\lambda(-q^{-1})\mathcal{X}_k^{\mu/\nu}(-q^{-1}). \tag{3}$$

We now calculate

$$\begin{aligned}
\left(\frac{1-q^{2nk}}{1-q^{2k}}\right)p_k\mathcal{G}_\nu(X;q) &= \left(\frac{1-q^{2nk}}{1-q^{2k}}\right)\sum_\mu c_\nu^\mu(q)p_k s_\mu \\
&= \left(\frac{1-q^{2nk}}{1-q^{2k}}\right)\sum_\mu c_\nu^\mu(q)\left(\sum_\lambda \chi_k^{\lambda/\mu} s_\lambda\right) \\
&= \sum_\lambda\left(\left(\frac{1-q^{2nk}}{1-q^{2k}}\right)\sum_\mu c_\nu^\mu(q)\chi_k^{\lambda/\mu}\right)s_\lambda \\
&= \sum_\lambda\left(\sum_{\mu\in\mathcal{P}_\delta} c_\mu^\lambda(q)\mathcal{X}_k^{\mu/\nu}(q)\right)s_\lambda \quad \text{using Equation (3)} \\
&= \sum_{\mu\in\mathcal{P}_\delta}\mathcal{X}_k^{\mu/\nu}(q)\mathcal{G}_\mu(X;q).
\end{aligned}$$

This proves the first statement. The second statement is proved in the same manner, considering B_{-k} instead of B_k. □

We shall see later in Section 2.9 that the lowering version of the Murnagham-Nakayama rule can be deduced combinatorially in a rather straightforward manner (using an observation of Schilling, Shimozono and White [SSW]). In fact it is clear that the lowering operator version is easier as the proof does not require the use of the commutator relation of B_k and B_{-k} in Theorem 3, only that the B_{-k} for $k \geq 1$ commute. Thus the lowering version of Theorem 23 is essentially equivalent to the fact that the $\mathcal{G}_\lambda(X;q)$ are symmetric.

Note that it is rather difficult to interpret Theorem 23 in terms of the n-quotient at $q = 1$. When $q = 1$ the product $\left(1 + q^{2k} + \cdots + q^{2k(n-1)}\right)p_k\mathcal{G}_\lambda(X;q)$ becomes

$$np_k s_{\lambda^{(0)}} s_{\lambda^{(1)}} \cdots s_{\lambda^{(n-1)}}$$

which may be written as the sum of n usual Murnagham-Nakayama rules as

$$\sum_{i=0}^{n-1} s_{\lambda^{(0)}} \cdots (p_k s_{\lambda^{(i)}}) \cdots s_{\lambda^{(n-1)}}.$$

Thus we might expect that border ribbon strips of size k correspond to adding a usual ribbon strip of size k to one partition in the n-quotient. However, the following example shows that this will not work.

Example 24. Let $n = 2$ and consider $(1+q^4)p_2 \cdot 1$. By the ribbon Murnagham-Nakyama rule ($\mathcal{G}_0 = 1$), this should equal to

$$\mathcal{G}_{(4)} + q\mathcal{G}_{(3,1)} + (q^2-1)\mathcal{G}_{(2,2)} - q\mathcal{G}_{(2,1,1)} - q^2\mathcal{G}_{(1,1,1,1)}.$$

We can compute directly that

$$\mathcal{G}_{(4)} = h_2, \quad \mathcal{G}_{(3,1)} = qh_2, \quad \mathcal{G}_{(2,1,1)} = qe_2$$
$$\mathcal{G}_{(2,2)} = q^2h_2 + e_2, \quad \mathcal{G}_{(1,1,1,1)} = q^2e_2,$$

verifying Theorem 23 directly. On the other hand, the shapes which correspond to a single border strip in one partition of the 2-quotient are $\{(4), (3,1), (2,1,1), (1,1,1,1)\}$ and the corresponding $\mathcal{G}_\lambda$ terms do not give $(1+q^4)p_2$.

It seems possible that the ribbon Murnagham-Nakayama rule may have some relationship with the representation theory of the wreath products $S_n\mathcal{S}C_p$, or even more likely to the cyclotomic Hecke algebras associated to these wreath products (see for example [Mat]).

2.4. The map $\mathbf{\Phi} : \mathbf{F} \to \Lambda(q)$

In this section we will study a linear map from $\mathbf{F}$ to $\Lambda(q)$.

Definition 25. Let $\mathbf{\Phi} : \mathbf{F} \to \Lambda(q)$ be the linear over $\mathbb{C}$ map defined by $q \mapsto -q^{-1}$ and

$$|\lambda\rangle \mapsto \mathcal{G}_\lambda.$$

As the $|\lambda\rangle$ form a basis of $\mathbf{F}$ it is clear that such a map exists and is unique. Since the $\mathcal{G}_\lambda$ span $\Lambda(q)$ but are not linearly dependent, this map is surjective but not injective. We would intuitively think that in doing so we have lost a lot of information by going from $\mathbf{F}$ to $\Lambda(q)$ but curiously this map has many remarkable properties. Note that the map $\mathbf{\Phi}$ should not be confused with the classical identification of $\mathbf{F}$ with Λ via $|\lambda\rangle \leftrightarrow s_\lambda$, which we shall comment about in Section 2.10.

The following theorem can be interpreted as saying that $\mathbf{\Phi}$ is a projection of $\mathbf{F}$ onto $\mathbb{C}(q)[H_-]$.

Theorem 26. *After changing q to $-q^{-1}$, the map $\mathbf{\Phi}$ is a map of H modules. More precisely, let A be an element of the Heisenberg algebra H. Identify $\mathbf{\Phi}(A)$ with the element of $\Lambda(q)$ given by*

$$B_k \mapsto k\frac{\partial}{\partial p_k}$$
$$B_{-k} \mapsto \left(\frac{1-q^{2nk}}{1-q^{2k}}\right) p_k.$$

(Recall from Section 1.2 that this identifies $\Lambda(q)$ with $\mathbb{C}(q)[H_-]$ and defines an action of H with $q \mapsto -q^{-1}$.) Then we have

$$\mathbf{\Phi}(A \cdot v) = \mathbf{\Phi}(A)\mathbf{\Phi}(v)$$

for any $v \in \mathbf{F}$.

Furthermore, we have

$$\mathbf{\Phi}(G_{n\lambda}) = s_\lambda[(1+q^2+\cdots+q^{2n-2})X].$$

Proof. The first claim follows from Theorem 23 and Proposition 6 as we can simply compare both sides of the equation on the spanning sets $|\lambda\rangle$ and $\mathcal{G}_\lambda$. To obtain the second claim, we apply the first claim with $v = |0\rangle$ and $A = S_\lambda$, using also $G_{n\lambda} = S_\lambda|0\rangle$ by Proposition 9. □

In later sections we shall see that the two involutions of $\mathbf{F}$, the bar involution and the involution $v \mapsto v'$ become algebra isomorphisms of $\mathbb{C}(q)[H_-] \cong \Lambda(q)$. While most of the later results can be phrased concisely in terms of the Heisenberg algebra, we shall continue to use symmetric function terminology, thinking of the $\mathcal{G}_\lambda(X;q)$ as elements in $\Lambda(q)$ rather than $\mathbb{C}(q)[H_-]$.

We also have the following explicit descriptions of $s_\lambda[(1+q^2+\cdots+q^{2n-2})X]$.

Corollary 27. *Let λ be a partition and δ a fixed n-core. In $\Lambda(q)$ we have*

$$s_\lambda[(1+q^2+\cdots+q^{2n-2})X] = \sum_\mu c^\lambda_\mu(q)\mathcal{G}_\mu(X;q)$$

and

$$s_\lambda[(1+q^2+\cdots+q^{2n-2})X] = \sum_{\mu,\nu} c^\lambda_\mu(q)c^\nu_\mu(q)s_\nu(X)$$

where the sums are over all partitions μ with n-core δ.

Proof. These are immediate consequences of Theorem 26 and Lemma 20 as $\mathbf{\Phi}(|\delta\rangle) = 1$ for an n-core δ. □

By Corollary 21 we have calculated the images of those global basis vectors which are highest weight vectors for $U_q(\widehat{\mathfrak{sl}}_n)$ in $\mathbf{F}$. It is not clear whether this leads to any interesting results concerning $G_{n\lambda}$ in $\mathbf{F}$.

Applying $\mathbf{\Phi}$ to both sides of Lemma 20 we obtain

Proposition 28. *Let λ and μ be partitions. Then*

$$s_\lambda[(1+q^2+\cdots+q^{2n-2})X]\mathcal{G}_\mu(X;q) = \sum_\nu c^\lambda_{\nu/\mu}(q)\mathcal{G}_\nu(X;q).$$

This could be thought of as some kind of Littlewood Richardson rule for ribbon tableaux. In fact the coefficients $c^\lambda_{\nu/\mu}(q)$ are the q-Littlewood Richardson coefficients which are the coefficients of the expansion of $\mathcal{G}_{\nu/\mu}(X;q)$ on the Schur basis. Of course there is no combinatorial description of these coefficients except in the case $n = 2$, via the Yamanouchi domino tableaux of Carré and Leclerc [CL].

2.5. Ribbon Pieri Formulae

Recall that in the classical theory of symmetric functions and in the enumerative geometry of the Grassmanian we have the formula

$$h_k s_\lambda = \sum_\mu s_\mu$$

where the sum is over all μ such that μ/λ is a horizontal strip with k boxes. This formula arises in the intersection theory of the Grassmanian as the intersection of a special Schubert class with an arbitrary Schubert class and was discovered geometrically by Pieri [Pie]. We will now give the ribbon analogue of the Pieri formula.

Let $n \geq 1$ be a fixed integer. Define the formal power series

$$H(t) = \prod_i \prod_{k=0}^{n-1} \frac{1}{1 - x_i q^{2k} t}$$

and

$$E(t) = \prod_i \prod_{k=0}^{n-1} \left(1 + x_i q^{2k} t\right).$$

As usual we may define symmetric functions $\mathbf{h}_k$ and $\mathbf{e}_k$ by

$$H(t) = \sum_k \mathbf{h}_k t^k$$

and

$$E(t) = \sum_k \mathbf{e}_k t^k.$$

Note that we have suppressed the integer n from the notation. We shall see later that the definitions of these power series are completely natural in the context of Robinson-Schensted ribbon insertion.

In plethystic notation, $\mathbf{h}_k = h_k[(1 + q^2 + \cdots + q^{(2n-2)})X]$ and $\mathbf{e}_k = e_k[(1 + q^2 + \cdots + q^{(2n-2)})X]$. This can be seen as follows. Write

$$\begin{aligned}
\log(\sum_k h_k t^k) &= \sum_i \log\left(\frac{1}{1 - x_i t}\right) \\
&= \sum_i \sum_r \frac{(x_i t)^r}{r} \\
&= \sum_r \frac{p_r t^r}{r}.
\end{aligned}$$

Now take the plethysm $\Upsilon_{q,n}(p_r) = (1 + q^{2r} + \cdots + q^{(2n-2)r})p_r$ and reverse all the steps.

The following theorem is an immediate consequence of Theorem 54 and Theorem 23. Alternatively, one could just use Theorem 26 and Proposition 5.

Theorem 29 (Ribbon Pieri Rule). *Let λ be a partition. Then*

$$\mathbf{h}_k \mathcal{G}_\lambda(X;q) = \sum_\mu q^{s(\mu/\lambda)} \mathcal{G}_\mu(X;q) \tag{4}$$

where the sum is over all partitions μ such that μ/λ is a horizontal n-ribbon strip with k ribbons. Here $s(\mu/\lambda)$ refers to the spin of the unique tableaux which is a horizontal ribbon strip of shape μ/λ. Also

$$\mathbf{e}_k\mathcal{G}_\lambda(X;q) = \sum_\mu q^{s(\mu/\lambda)}\mathcal{G}_\mu(X;q)$$

where the sum is over all partitions μ such that μ/λ is a vertical n-ribbon strip with k ribbons. Here $s(\mu/\lambda)$ refers to the spin of the unique tableaux which is a vertical ribbon strip of shape μ/λ.

One can obviously obtain the corresponding formulae for $\mathbf{h}_\alpha = \mathbf{h}_{\alpha_1}\cdots\mathbf{h}_{\alpha_r}$ in terms of ribbon tableaux with weight α.

We can also obtain the two statements of Theorem 29 from each other via the involution ω_n of Section 2.6. Note that by Theorem 29, we have

$$\mathbf{h}_k = \sum_\lambda q^{\mathrm{mspin}(\lambda)}\mathcal{G}_\lambda(X;q)$$

where the sum is over all λ with no n-core such that $|\lambda| = kn$ with no more than n rows. Applying the usual Cauchy identity one sees that

$$\mathbf{h}_k = \sum_{|\mu|=k} s_\mu(1,q^2,\ldots,q^{2(n-1)})s_\mu(X).$$

Taking the coefficient of p_1^n on both sides we see that a modified spin generating function of ribbon tableaux T of size k and shape λ satisfying $\tilde{\lambda} = \emptyset$ and $l(\lambda) \leq n$ is

$$\sum_T q^{\mathrm{mspin}(sh(T))}q^{s(T)} = \sum_{|\mu|=k} s_\mu(1,q^2,\ldots,q^{2(n-1)})f^\mu$$

where f^μ denotes the number of standard Young tableaux of shape μ.

Example 30. Let $n=3$, $k=2$ and $\lambda=(3,1)$. Then

$$\mathbf{h}_2\mathcal{G}_{(3,1)} = \mathcal{G}_{(9,1)} + q\mathcal{G}_{(6,2,2)} + q^2\mathcal{G}_{(4,4,2)} + q^2\mathcal{G}_{(6,1,1,1,1)} + q^3\mathcal{G}_{(3,3,2,1,1)} + q^4\mathcal{G}_{(3,2,2,2,1)}.$$

Setting $q=1$ in $H(t)$ we see that

$$\mathbf{h}_k(X;1) = \sum_\alpha h_\alpha$$

where the sum is over all compositions $\alpha = (\alpha_0,\ldots,\alpha_{n-1})$ satisfying $\alpha_0+\cdots+\alpha_{n-1} = k$. We may thus interpret Theorem 29 at $q=1$ in terms of the n-quotient as the following formula:

$$\left(\sum_\alpha h_\alpha\right)s_{\lambda^{(0)}}\cdots s_{\lambda^{(n-1)}} = \sum_\alpha \left(h_{\alpha_0}s_{\lambda^{(0)}}\right)\cdots\left(h_{\alpha_{n-1}}s_{\lambda^{(n-1)}}\right) \tag{5}$$

where the sum is over the same set of compositions as above. Note that the right hand side of (5) is indeed equal to the right hand side of (4) at $q=1$ since a horizontal ribbon strip of size k is just a union of horizontal strips with total size k in the n-quotient.

It is clear that we also obtain lowering versions of the Pieri rules. If $h_k = f(p_1, p_2, \ldots)$ we know that the adjoint operator (with respect to the usual inner product) is $h_k^\perp = f(\frac{\partial}{\partial p_1}, 2\frac{\partial}{\partial p_2}, \ldots)$. Thus by Theorem 23 and Theorem 54 we have

Proposition 31 (Ribbon Pieri Rule – Lowering Version)**.** *Let λ be a partition and $k \geq 1$ be an integer. Then*

$$h_k^\perp \mathcal{G}_\lambda(X;q) = \sum_\mu q^{s(\lambda/\mu)} \mathcal{G}_\mu(X;q)$$

where the sum is over all μ such that λ/μ is a horizontal ribbon strip and $s(\lambda/\mu)$ is the spin of such a horizontal ribbon strip. Similarly,

$$e_k^\perp \mathcal{G}_\lambda(X;q) = \sum_\mu q^{s(\lambda/\mu)} \mathcal{G}_\mu(X;q)$$

where the sum is over all μ such that λ/μ is a vertical ribbon strip and $s(\lambda/\mu)$ is the spin of such a vertical ribbon strip.

This is a spin version of a branching formula first observed by Schilling, Shimozono and White [SSW] (see Section 2.9).

2.6. The Ribbon Involution ω_n

In this section we will define an involution w_n on $\Lambda(q)$ which is essentially the involution $v \mapsto v'$ on the Fock space $\mathbf{F}$ of Section 1.3. However, this involution will turn out to be not just a semi-linear involution, but also a $\mathbb{C}$-algebra isomorphism of $\Lambda(q)$.

Definition 32. Define the *ribbon involution* $w_n : \Lambda(q) \to \Lambda(q)$ as the semi-linear map satisfying $w_n(q) = q^{-1}$ and

$$w_n(s_\lambda) = q^{(n-1)|\lambda|} s_{\lambda'}.$$

Theorem 33. *The map w_n is an $\mathbb{C}$-algebra homomorphism which is an involution. It maps $\mathcal{G}_{\lambda/\mu}$ into $\mathcal{G}_{(\lambda/\mu)'}$ for every skew shape λ/μ.*

Proof. The fact that w_n is an algebra homomorphism follows from the fact that if $s_\lambda s_\mu = \sum c^\nu_{\lambda\mu} s_\nu$ then $s_{\lambda'} s_{\mu'} = \sum c^\nu_{\lambda\mu} s_{\nu'}$, and that the grading is preserved by multiplication. That w_n is an involution is a quick calculation.

For the last statement, we use Proposition 11 and Lemma 20 which give

$$(S_\nu|\mu\rangle)' = (-q)^{(n-1)k} S_{\nu'}|\mu'\rangle$$
$$\sum_\lambda c^\nu_{\lambda/\mu}(-q)|\lambda'\rangle = (-q)^{(n-1)k} \sum_\lambda c^{\nu'}_{\lambda'/\mu'}(-q^{-1})|\lambda'\rangle.$$

Here $k = |\nu|$. Equating coefficients of $|\lambda'\rangle$ and changing q to $-q^{-1}$ we obtain

$$c^\nu_{\lambda/\mu}(q^{-1}) = q^{-(n-1)k} c^{\nu'}_{\lambda'/\mu'}(q).$$

Thus

$$
\begin{aligned}
w_n(\mathcal{G}_{\lambda/\mu}) &= \sum_{\nu} w_n(c^{\nu}_{\lambda/\mu}(q)s_{\nu}) \\
&= \sum_{\nu} \left(c^{\nu'}_{\lambda'/\mu'}(q)q^{-(n-1)|\nu|}\right) q^{(n-1)|\nu|} s_{\nu'} \\
&= \mathcal{G}_{\lambda'/\mu'}.
\end{aligned}
$$

□

Proposition 34. *Let $f \in \Lambda(q)$ have degree k. Then we have*

$$q^{2(n-1)k}\omega_n\left(\Upsilon_{q,n}(f)\right) = \Upsilon_{q,n}\left(\omega_n(f)\right).$$

In particular,

$$\omega_n\left(s_\lambda[(1+q^2+\cdots+q^{2(n-1)})X]\right) = q^{-(n-1)k}s_\lambda[(1+q^2+\cdots+q^{2(n-1)})X].$$

Proof. Since both ω_p and $\Upsilon_{q,n}(f)$ are $\mathbb{C}$-algebra homomorphisms we need only check this for the elements p_k and for q, for which the computation is straightforward. □

2.7. The Ribbon Cauchy Identity

It is well known (see [Mac, EC2]) that the Schur functions satisfy the equation

$$\sum_{\lambda \in \mathcal{P}} s_\lambda(X)s_\lambda(Y) = \prod_{i,j} \frac{1}{1-x_iy_j} \tag{6}$$

known as the Cauchy identity. This is equivalent to the fact that the Schur functions form an orthonormal basis of Λ. It also describes the decomposition of the polynomial functions on the $m \times m$ matrices under the (commuting) left and right actions of GL_m. Equation (6) is often proven combinatorially via the Knuth's extension of the Robinson-Schensted correspondence, which is a bijection between matrices with certain row and column sums and pairs of semistandard Young tableaux of the same shape. This combinatorial approach to the Cauchy identity for ribbon tableaux will be studied in Section 3.1.

Let us write the formal power series

$$\Omega(X;q) = \prod_{i,j}\prod_{k=0}^{n-1} \frac{1}{1-x_iy_jq^{2k}}.$$

A dual version of this series is

$$\tilde{\Omega}(X;q) = \prod_{i,j}\prod_{k=0}^{n-1}\left(1+x_iy_jq^{2k}\right).$$

Theorem 35 (Ribbon Cauchy Identity)**.** *Fix n as usual and a n-core δ. Then*

$$\Omega(X;q) = \sum \mathcal{G}_\lambda(X;q)\mathcal{G}_\lambda(Y;q)$$

where the sum is over all λ such that $\tilde{\lambda} = \delta$.

Note that this does not imply that the $\mathcal{G}_\lambda$ form an orthonormal basis under a certain inner product, as they are not linearly independent.

Proof. By Corollary 27 we have

$$s_\lambda[(1+q^2+\cdots+q^{2n-2})X] = \sum_\mu c_\mu^\lambda(q)\mathcal{G}_\mu(X;q)$$

where the sum is over all $\mu \in \mathcal{P}_\delta$. Thus

$$\begin{aligned}\sum_\lambda s_\lambda[(1+q^2+\cdots+q^{2n-2})X]s_\lambda(Y) &= \sum_\mu \left(\sum_\lambda c_\mu^\lambda(q)s_\lambda(Y)\right)\mathcal{G}_\mu(X;q) \\ &= \sum_\mu \mathcal{G}_\mu(X;q)\mathcal{G}_\mu(Y;q).\end{aligned}$$

Let $\Upsilon_{q,n}(X)$ denote the algebra automorphism of $\Lambda[X](q) \otimes_{\mathbb{C}(q)} \Lambda[Y](q)$ given by

$$p_k(X) \mapsto (1+q^{2k}+\cdots+q^{(2n-2)k})p_k(X).$$

Applying $\Upsilon_{q,n}(X)$ to

$$\log\left(\prod_{i,j}\frac{1}{1-x_iy_j}\right) = \sum_k \frac{1}{n}p_k(X)p_k(Y)$$

gives

$$\log\left(\prod_{i,j}\prod_{k=1}^{n-1}\frac{1}{1-x_iy_jq^{2k}}\right)$$

which is exactly $\log(\Omega)$. But applying $\Upsilon_{q,n}(X)$ to the left hand side of (6) gives

$$\sum_\lambda s_\lambda[(1+q^2+\cdots+q^{2n-2})X]s_\lambda(Y)$$

from which the Theorem follows. $\square$

Now let us compute $\omega_n(\Omega)$ where we let

$$\omega_n : \Lambda[X](q) \otimes_{\mathbb{C}(q)} \Lambda[Y](q) \to \Lambda[X](q) \otimes_{\mathbb{C}(q)} \Lambda[Y](q)$$

act on the X variables by

$$\omega_n(f(X;q)\otimes g(Y;q)) \mapsto \omega_n(f(X;q)) \otimes g(Y;q^{-1}).$$

One checks immediately that this is indeed an algebra involution. We have (fixing an n-core δ)

$$\begin{aligned}\omega_n(\Omega) &= \omega_n\left(\sum_{\lambda\in\mathcal{P}_\delta}\mathcal{G}_\lambda(X;q)\mathcal{G}_\lambda(Y;q)\right) \\ &= \sum_{\lambda\in\mathcal{P}_\delta}\mathcal{G}_{\lambda'}(X;q)\mathcal{G}_\lambda(Y;q^{-1}).\end{aligned}$$

Also,

$$\begin{aligned}\omega_n(\Omega) &= \omega_n\left(\sum_\lambda s_\lambda(X)s_\lambda[(1+q^2+\cdots+q^{2(n-1)})Y]\right)\\ &= \sum_\lambda q^{(n-1)|\lambda|}s_{\lambda'}(X)s_\lambda[(1+q^{-2}+\cdots+q^{-2(n-1)})Y]\\ &= \prod_{i,j}\prod_{k=0}^{n-1}(1+x_iy_jq^{n-1-2k}).\end{aligned}$$

Thus

$$\sum_{\lambda\in\mathcal{P}_\delta}\mathcal{G}_{\lambda'}(X;q)\mathcal{G}_\lambda(Y;q^{-1}) = \prod_{i,j}\prod_{k=0}^{n-1}(1+x_iy_jq^{n-1-2k}).$$

If we multiply the d^{th} graded piece of each side by $q^{(n-1)d}$ we obtain the following result.

Proposition 36. *Fix an n-core δ. We have*

$$\tilde{\Omega} = \sum_{\lambda\in\mathcal{P}_\delta} q^{(n-1)|\lambda/\tilde{\lambda}|}\mathcal{G}_{\lambda'}(X;q)\mathcal{G}_\lambda(Y;q^{-1}).$$

The factor of $q^{(n-1)|\lambda/\tilde{\lambda}|}$ can be explained combinatorially by the fact that $s(T') = q^{(n-1)|\lambda/\tilde{\lambda}|} - s(T)$ for a ribbon tableaux T and its conjugate T' satisfying $sh(T) = \lambda$.

2.8. The Ribbon Inner Product and the Bar Involution on $\Lambda(q)$

In this section we will define an inner product on $\Lambda(q)$ which seems particularly adapted to the study of ribbon functions. We will also give a $\mathbb{C}$-algebra involution on $\Lambda(q)$ which is compatible with the bar involution of $\mathbf{F}$ (for at least the space of highest weight vectors).

Definition 37. Let $\langle .,. \rangle_n : \Lambda(q)\times\Lambda(q)\to\mathbb{C}(q)$ be the $\mathbb{C}(q)$-bilinear map defined by

$$\langle \Upsilon_{q,n}(p_\lambda), p_\mu\rangle = z_\lambda\delta_{\lambda\mu}.$$

It is clear that $\langle .,. \rangle_n$ is non-degenerate.

The inner product $\langle .,. \rangle_n$ is related to Ω in the same way as the usual inner product is related to the usual Cauchy kernel:

Proposition 38. *Two bases $\{v_\lambda\}$ and $\{w_\lambda\}$ of $\Lambda(q)$ are dual with respect to $\langle\, .\, \rangle_n$ if and only if*

$$\sum_\lambda v_\lambda(X)w_\lambda(Y) = \Omega.$$

In particular, $\left\{s_\lambda[(1+q^2+\ldots+q^{2(n-1)})X]\right\}$ is dual to $\{s_\lambda\}$.

Proof. It is clear that $\{p_\lambda/z_\lambda\}$ and $\{\Upsilon_{q,n}p_\lambda\}$ are dual. But applying $\Upsilon_{q,n}$ to the usual Cauchy kernel gives

$$\sum_\lambda \frac{1}{z_\lambda}\Upsilon_{q,n}(p_\lambda(X))p_\lambda(Y) = \Omega.$$

To see that this is true for any pair of dual bases of $\Lambda(q)$ with respect to $\langle \,.,\,. \rangle_n$ is an exercise in linear algebra (see for example [EC2, Lemma 7.9.2]). The last statement is a consequence of Theorem 35. $\square$

In fact if $\{v_\lambda\}$ and $\{w_\lambda\}$ are dual basis of Λ with respect to the usual inner product then it is clear that $\{\Upsilon_{q,n}(v_\lambda)\}$ and $\{w_\lambda\}$ are dual with respect to $\langle \,.,\,. \rangle_n$. We now give some basic properties of $\langle \,.,\,. \rangle_n$.

Lemma 39. *The inner product $\langle \,.,\,. \rangle_n$ is symmetric.*

Proof. This is clear from the definition as we can just check this on the basis p_λ of $\Lambda(q)$. $\square$

Recall that for $f \in \Lambda$, $f^\perp$ denotes its adjoint with respect to the usual inner product.

Proposition 40. *The operator $f^\perp$ is adjoint to multiplication by $\Upsilon_{q,n}(f) \in \Lambda(q)$.*

Proof. This is a consequence of $\langle f, g\rangle = \langle \Upsilon_{q,n}(f), g\rangle_n$. $\square$

The inner product $\langle \,.,\,. \rangle_n$ is compatible with the inner product $\langle |\lambda\rangle, |\mu\rangle\rangle = \delta_{\lambda\mu}$ on $\mathbf{F}$ when we restrict our attention to the space of highest weight vectors. In $\mathbf{F}$ we have $\langle B_k u, v\rangle = \langle u, B_{-k}v\rangle$ for any $u, v \in \mathbf{F}$, see [LT, Proposition 7.9] which corresponds to Proposition 40.

The bar involution $^- : \mathbf{F} \to \mathbf{F}$ of Section 1.3 also has an image under $\mathbf{\Phi}$.

Definition 41. Define the $\mathbb{C}$-algebra involution $^- : \Lambda(q) \to \Lambda(q)$ by $\overline{q} = q^{-1}$ and

$$p_k \mapsto q^{2(n-1)k}p_k.$$

It is clear that $^-$ is indeed an involution. The following proposition shows in particular that $^- : \Lambda(q) \to \Lambda(q)$ is the image of the bar involution on $\mathbf{F}$ under $\mathbf{\Phi}$. This implies that $\langle \mathbf{\Phi}(u), \mathbf{\Phi}(v)\rangle_n = \langle u, v\rangle$ for two vectors $u, v \in \mathbf{F}$ lying within the subspace of highest weight vectors.

Proposition 42. *Let $u, v \in \Lambda(q)$. The involution $^- : \Lambda(q) \to \Lambda(q)$ has the following properties:*

$$\begin{aligned} \overline{\mathbf{\Phi}(G_{n\lambda})} &= \mathbf{\Phi}(G_{n\lambda}), \\ \overline{\Upsilon_{q,n}(p_k)} &= \Upsilon_{q,n}(p_k), \\ \langle \overline{u}, v\rangle_n &= \left\langle \omega_n(u), \overline{\omega_n(v)} \right\rangle. \end{aligned}$$

Proof. As $^-$ is an algebra homomorphism, the first statement follows from the second statement and Theorem 26. The second statement is a straightforward computation. For the last statement, we compute explicitly both sides for the basis p_λ of $\Lambda(q)$. $\square$

Proposition 42 shows that $\overline{\mathbf{\Phi}(v)} = \mathbf{\Phi}(\overline{v})$ for all u, v in the subspace of highest weight vectors in $\mathbf{F}$. However this is not true in general. For example, $|(3,1)\rangle + q|(2,2)\rangle + q^2|(2,1,1)\rangle$ is bar invariant in $\mathbf{F}$ but its image under $\mathbf{\Phi}$ is not.

2.9. Skew and super ribbon functions

We now describe some properties of the skew ribbon functions $\mathcal{G}_{\lambda/\mu}(X;q)$. Unfortunately, we have been unable to describe them in terms of an adjoint in analogy with the formula

$$s_{\lambda/\mu} = s_\lambda^\perp s_\mu.$$

However, the following proposition is an analogue of

$$\sum_\lambda s_\lambda(X) s_{\lambda/\mu}(Y) = s_\mu(X) \prod_{i,j} \frac{1}{1 - x_i y_j}.$$

Proposition 43. *Let μ be any partition. Then*

$$\mathcal{G}_\mu(X;q)\Omega = \sum_\lambda \mathcal{G}_\lambda(X;q)\mathcal{G}_{\lambda/\mu}(Y;q)$$

where the sum is over all λ satisfying $\tilde{\lambda} = \tilde{\mu}$.

Proof. Lemma 20 implies that

$$\Upsilon_{q,n}(s_\nu(X))\mathcal{G}_\mu(X;q) = \sum_\lambda c^\nu_{\lambda/\mu}(q)\mathcal{G}_\lambda(X;q).$$

Now multiply both sides by $s_\nu(Y)$ and sum over ν. Finally use Theorem 35. □

Another essentially equivalent way in which skew ribbon functions arise was observed by Schilling, Shimozono and White [SSW] in cospin form. By the combinatorial definition of $\mathcal{G}_\lambda$ we immediately have the coproduct expansion

$$\mathcal{G}_\lambda(X+Y;q) = \sum_\mu \mathcal{G}_\mu(X;q)\mathcal{G}_{\lambda/\mu}(Y;q).$$

Since ([Mac])

$$\Delta f = \sum_\mu s_\mu^\perp f \otimes s_\mu$$

we get immediately that

$$s_\nu^\perp \mathcal{G}_\lambda(X;q) = \sum_\mu \mathcal{G}_\mu(X;q)\left\langle \mathcal{G}_{\lambda/\mu}(Y;q), s_\nu \right\rangle.$$

Setting $\nu = (k)$ we obtain the lowering version of the Pieri rule (Proposition 31):

$$h_k^\perp \mathcal{G}_\lambda(X;q) \sum_\mu q^{s(\lambda/\mu)}\mathcal{G}_\mu(X;q)$$

where the sum is over all μ such that λ/μ is a horizontal ribbon strip of size k.

We would like to mention another generalisation of the usual ribbon functions which are super ribbon functions. Fix a total order on two alphabets $A = \{1 < 2 < 3 < \cdots\}$

and $A' = \{1' < 2' < 3' < \cdots\}$ (which we assume to be compatible with each of their natural orders).

Definition 44. A super ribbon tableaux T of shape λ/μ is a ribbon tableaux of the same shape with ribbons labelled by the two alphabets such that the skew shape containing ribbons labelled $a \in A$ form a horizontal ribbon strip and those labelled $a' \in A'$ form a vertical ribbon strip. These strips are required to be compatible with the chosen total order on $A \cup A'$, as usual.

Define the *super ribbon function* $\mathcal{G}_{\lambda/\mu}(X/Y;q)$ as the following weight and spin generating function:

$$\mathcal{G}_{\lambda/\mu}(X/Y;q) = \sum_T q^{s(T)} \mathbf{x}^{w(T)} \mathbf{y}^{w'(T)}$$

where the sum is over all super ribbon tableaux T of shape λ/μ and $w(T)$ is the weight in the first alphabet A while $w'(T)$ is the weight in the second alphabet A'.

Proposition 45. *The super ribbon function $\mathcal{G}_{\lambda/\mu}(X/Y;q)$ is a symmetric function in the X and Y variables, separately.*

Proof. The proof is completely analogous to that of Theorem 14, using the commutativity of both the operators $\mathcal{V}_k$ and $\tilde{\mathcal{V}}_k$. □

No doubt the super ribbon functions can be studied in the same way that super Schur functions are.

2.10. Open questions and other aspects of ribbon functions

In this section we describe some other aspects of ribbon functions which we have not mentioned or which may be interesting for further study. The problem of finding combinatorial proofs of the theorems in this part will be addressed in Section 3.1.

Cospin vs. spin. Recall that $\mathrm{cosp}(T) = \mathrm{mspin}(T) - s(T)$ for a ribbon tableaux T. It is easy to see that $\mathrm{cosp}(T)$ is always even. In many situations it appears that the statistic cospin is more natural than the statistic spin. For example, Lascoux, Leclerc and Thibon [LLT] have shown that the cospin $\tilde{\mathcal{H}}(X;q)$ functions are generalisations of Hall-Littlewood Functions. Cospin also appears to be the natural statistic when finding connections between ribbon tableaux and rigged configurations (see for example [Sch]).

All the formulae in this Part can be phrased in terms of cospin if suitable powers of q are inserted. However, it is clear that the formulae presented in terms of spin is the more natural form.

Vertex Operators. The relationship between the ribbon functions and the Heisenberg algebra suggests that we may want to study the affect of the 'vertex operators'

$$A_r = \sum_i \mathbf{h}_{r-i} \mathbf{e}_i^{\perp}$$

on the ribbon functions. It is well known that both the Schur functions and the Hall Littlewood functions [Jin] can be developed in the context of these vertex operators so perhaps the ribbon functions can be studied in the same way.

Generalised Kostka polynomials. There is a mysterious and unsolved connection between the generalised Kostka polynomials of [KS, SW3, SchW] and the q-Littlewood Richardson coefficients. The coefficients of the generalised Kostka polynomials are not always positive but when the n-quotient is a sequence of rectangles the q-Littlewood Richardson coefficients appear to coincide with the generalised Kostka polynomials. A vertex operator description of the generalised Kostka polynomials has also been given by Shimozono and Zabrocki [SZ]. Perhaps the vertex operators described above will help.

Jacobi-Trudi and alternant formulae. The Pieri rule (Theorem 29) we have given stops short of giving a closed formula for the functions $\mathcal{G}_\lambda(X;q)$. It is well known that the Schur functions can be written as

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}$$

and as

$$s_\lambda = det(h_{\lambda_i - i + j})_{i,j=1}^{l(\lambda)}.$$

Most algebraic treatments (see [Mac]) of the theory of symmetric functions use these formulae as the basis of all the algebraic computations for Schur functions. It would be nice to have a similar closed formula for the ribbon functions.

Enumerative problems. Stanley [EC2] has given a 'hook content formula' for the specialisation $s_\lambda(1, t, t^2, \ldots, t^r)$ of the Schur functions. In particular this gives the hook length formula for the number of standard Young tableaux of a particular shape. At $q = 1$ the corresponding problem for ribbon tableaux is trivial due to Littlewood's n-quotient map. However, can anything be done for arbitrary q?

When $n = 2$, the specialisation $q^2 = -1$ relates domino tableaux to the study of enumerative study of sign-imbalance [Sta, Whi, Lam]. It is not clear whether this can be generalised to arbitrary n.

Graded S_n representations and $\mathbf{h}_k$. The non-negativity of the q-Littlewood Richardson coefficients $c^\mu_\lambda(q)$ would follow from the existence of a graded S_n representation with Frobenius character $\mathcal{G}_\lambda(X;q)$ (where the coefficient of powers of q correspond to the graded parts).

Such a graded S_n representation can be easily described for the ribbon homogeneous function $\mathbf{h}_k$ (see [EC2, Ex. 7.75]). Let S_k act on the multiset $M = \{1^{n-1}, 2^{n-1}, \ldots, k^{n-1}\}$ in the natural way. Then the representation we seek is given by the action of S_k on the subsets of M with the grading given by the size of such a subset. This suggests that one might seek subrepresentations of this representation which correspond to the $\mathcal{G}_\lambda(X;q)$ for $l(\lambda) \leq n$.

Connections with diagonal harmonics. When $k = n-1$, the function $\mathbf{e}_k$ also appears in work of Garsia and Haiman [GH] on the bigraded character $DH_n(X;q,t)$ of the diagonal harmonics. More specifically, we have

$$DH_n(X;q,1/q)q^{\binom{n}{2}}[n+1]_q = e_n[(1+q+\ldots+q^n)X].$$

Curiously, ribbon functions also appear in recent work of [HHLRU]. Perhaps a deeper relationship between exists.

Other generating functions. In [KLLT], Kirillov, Lascoux, Leclerc and Thibon gave a number of generating functions for domino functions which were subsequently generalised in [Lam]. As a special case, we have the following product expansion for $n = 2$:

$$\sum_{\lambda} \mathcal{G}^{(2)}_{\lambda}(X;q) = \frac{\prod_i (1+qx_i)}{\prod_i(1-x_i)\prod_i(1-q^2x_i^2)\prod_{i<j}(1-x_ix_j)\prod_{i<j}(1-q^2x_ix_j)}.$$

Can this be generalised to other values of n?

Other incarnations of Λ. Often the Fock space $\mathbf{F}$ is identified with Λ via

$$|\lambda\rangle \leftrightarrow s_\lambda.$$

This gives $\mathbf{F}$ the extra structure of an algebra. In this context, our map $\mathbf{\Phi}$ can be considered to be an operator from $\Lambda(q)$ to $\Lambda(q)$. In the notation of [LLT], $\mathbf{\Phi}$ would be the adjoint ϕ_q of the operator p_n^q which sends h_α to $\mathcal{V}_\alpha \cdot 1$. It is not clear whether this point of view leads to more results.

Leclerc [Lec] has studied another embedding $\iota : \Lambda \to \mathbf{F}$ given by

$$p_\lambda \mapsto B_{-\lambda}|0\rangle.$$

Altering this slightly, we may define a $\mathbb{C}(q)$-linear embedding $\iota_q : \Lambda(q) \to \mathbf{F}$ by

$$\Upsilon_{q,n}(p_\lambda) \mapsto B_{-\lambda}|0\rangle.$$

By Theorem 26, we see that the composition

$$\mathbf{\Phi} \circ \iota_q : \Lambda(q) \to \Lambda(q)$$

is the identity. Leclerc has connected ι with the Macdonald polynomials and it is likely that our setup can be connected with many other aspects of symmetric function theory in this way.

Part 3. **Combinatorics**

3.1. Ribbon Insertion

In this section we put the ribbon Pieri formula (Theorem 29) and ribbon Cauchy identity (Theorem 35) in the context of ribbon Robinson-Schensted-Knuth (RSK) insertion, where both will be proven completely for the case $n = 2$. Note that we will discuss everything in the special case of an empty n-core, though everything can be generalised to the non-empty n-core case.

3.1.1. **Robinson-Schensted-Knuth for usual Young tableaux.** Recall that the usual Robinson-Schensted bijection gives a bijection between permutations $w \in S_m$ and pairs of standard Young tableaux (see [EC2]):

$$w \mapsto (P(w), Q(w)).$$

The tableaux $P(w)$ is defined recursively by the insertion algorithm. The tableaux $(T \leftarrow i)$ is obtained from T by placing i in the leftmost square of the first row of T where the placement is legal (i is placed legally if it is greater than all numbers preceding it). If this displaces an element j then the element j is placed in the

leftmost square of T where the placement is legal. This continues until no more bumping occurs. Then the insertion tableaux $P(w)$ is defined as

$$P(w) = ((\cdots((\emptyset \leftarrow w_1) \leftarrow w_2)\cdots) \leftarrow w_n).$$

The recording tableaux $Q(w)$ is defined by requiring that

$$sh(Q(w)|_i) = sh((\cdots((\emptyset \leftarrow w_1) \leftarrow w_2)\cdots) \leftarrow w_i),$$

where $Q(w)|_i$ is the subtableaux of $Q(w)$ obtained by erasing all the squares with numbers greater than i.

This bijection generalises to a bijection between two line arrays

$$w = \begin{pmatrix} i_1 & \cdots & i_m \\ j_1 & \cdots & j_m \end{pmatrix}$$

satisfying (a) $i_1 \leq i_2 \leq \cdots \leq i_m$ and (b) if $i_r = i_{r+1}$ then $j_r \leq j_{r+1}$. The weight of the top row then becomes the weight of the tableaux $Q(w)$ while the weight of the bottom row becomes the weight of the tableaux $P(w)$. This immediately leads to the Cauchy identity (6) as the generating function (where a number i on the top row has weight y_i and a number j on the borrom row has weight x_j) for such two line arrays is exactly

$$\prod_{i,j} \frac{1}{1 - x_i y_j}.$$

To obtain the bijection $w \mapsto (P(w), Q(w))$ one needs an important property of insertion. Let

$$T' = (T \leftarrow i)$$
$$T'' = (T' \leftarrow j)$$

denote the the result from two successive insertions, and let $\gamma = sh(T'/T)$ and $\theta = sh(T''/T)$ be the two new squares added to the shapes. Then γ lies to the left of θ if and only if $i \leq j$. This *increasing insertion* property guarantees that $Q(w)$ will be semistandard. In fact it is this property that is crucial to a combinatorial proof (see [EC2, p. 341]) of the Pieri rule:

$$h_k s_\lambda = \sum_\mu s_\mu.$$

We may interpret h_k as the generating function for a k-tuple of increasing positive integers ($i_1 \leq i_2 \leq \cdots \leq i_k$), and s_λ as the weight generating function of tableaux T with shape λ, as usual. Then a bijection from the left hand side to the right hand side is obtained by associating to a pair $((i_1, \cdots, i_k), T)$ the tableaux

$$T' = ((\cdots((T \leftarrow i_1) \leftarrow i_2)\cdots) \leftarrow i_k).$$

The increasing insertion property guarantees that $sh(T')/\lambda$ is indeed a horizontal strip.

3.1.2. **General ribbon insertion.** Following [SW2], we will call a three line array $\mathbf{w}$ a n-colored biword if it is of the form

$$\mathbf{w} = \begin{pmatrix} c_1 & \cdots & c_m \\ i_1 & \cdots & i_m \\ j_1 & \cdots & j_m \end{pmatrix}$$

where the c_i are the 'colors' taking values in $\{0, \ldots, n-1\}$ and the i_k, j_k are positive integers. We will insist each such colored biword to have a canonical ordering in such a way that only the multiset of ordered triples $\{(c_k, i_k, j_k)\}$ matters. For example, we could choose the lexicographic ordering so (a) $c_k \leq c_{k+1}$ and (b) if $c_k = c_{k+1}$ then $i_k \leq i_{k+1}$ and (c) if $c_k = c_{k+1}$ and $i_k = i_{k+1}$ then $j_k \leq j_{k+1}$. Giving the weight

$$w((c_k, i_k, j_k)) = q^{2c_k} y_{i_k} x_{j_k}$$

to each triple, the weight generating function of colored biwords becomes

$$\prod_{i,j} \prod_{k=0}^{n-1} \frac{1}{1 - x_i y_j q^k}.$$

Note that when $n = 1$ we just recover the setup of the previous subsection. We are then led to the following observation.

Observation 46. *A (ribbon RSK) bijection $\pi : \mathbf{w} \mapsto (P_r(\mathbf{w}), Q_r(\mathbf{w}))$ between* colored biwords *and pairs of ribbon tableaux of the same shape and fixed n-core will prove the Cauchy formula of Theorem 35 if*

- *The bijection π is weight preserving. Thus the weight of the second line of $\mathbf{w}$ is $w(Q_r(\mathbf{w}))$ and the weight of the third line of $\mathbf{w}$ is $w(P_r(\mathbf{w}))$.*
- *The bijection π sends color to spin. Thus*

$$2(c_1 + c_2 + \ldots + c_m) = s(P_r(\mathbf{w})) + s(Q_r(\mathbf{w})). \tag{7}$$

Suppose now that the bijection π is defined recursively via insertion of ribbons (c, j) into a tableaux T:

$$P_r(\mathbf{w}) = ((\cdots((\emptyset \leftarrow (c_1, j_1)) \leftarrow (c_2, j_2)) \cdots) \leftarrow (c_m, j_m)).$$

Then the tableaux $T' = T \leftarrow (c, j)$ should satisfy (a) the tableaux T' has an extra ribbon labelled j, (b) $sh(T')/sh(T)$ is a ribbon, and (c) $s(T') + s(sh(T')/sh(T)) = s(T) + 2c$. Here we will think of (c, j) as a ribbon labelled j with spin c. Let

$$T' = T \leftarrow (c, j)$$
$$T'' = T' \leftarrow (c', j').$$

A *ribbon increasing insertion* property is a property of the form

> The ribbon $sh(T')/sh(T)$ lies to the left of $sh(T'')/sh(T')$ if and only if $(c, j) \leq (c', j')$.

Here $<$ should be some total order on ribbons labelled j with spin c. Fix a ribbon tableaux T. Then we can construct a bijection between sets of ribbons $\{(c_i, j_i)\}$ and ribbon tableaux T' whose shape differs from that of T by a horizontal ribbon strip by

$$T' = ((\cdots((T \leftarrow (c_1, j_1)) \leftarrow (c_2, j_2))\cdots) \leftarrow (c_k, j_k)).$$

The ribbons (c_i, j_i) are inserted according to the order $<$ thus ensuring the resulting shape changes by a horizontal ribbon strip. Thus:

Observation 47. *A* ribbon increasing insertion *property for π leads to a combinatorial proof of the ribbon Pieri formula (Theorem 29).*

Thus the generating function $H(t)$ of Section 2.5 can be interpreted as the generating functions of ribbons (c, j) with weight $w(c, j) = q^{2k}x_j$.

3.1.3. **Domino insertion.** The above comments become proofs for the case $n = 2$. Barbasch and Vogan [BV] have defined domino insertion in connection with the primitive ideals of classical lie algebras. This was put into the usual bumping description by Garfinkle [Gar]. Recently, Shimozono and White [SW] have extended Garfinkle's description to the semistandard case and connected it with mixed insertion. They also observed that it had the crucial color-to-spin property. A straightforward extension to the non-empty 2-core case was presented in [Lam]. We thus have:

Theorem 48. *Fix a 2-core δ. There is a bijection between colored biwords $\mathbf{w}$ of length m with two colors $\{0, 1\}$ and pairs $(P_d(\mathbf{w}), Q_d(\mathbf{w}))$ of semistandard domino tableaux with the same shape $\lambda \in \mathcal{P}_\delta$ and $|\lambda| = 2m + |\delta|$ with the following properties:*

(1) *The bijection has the color-to-spin property:*

$$tc(\mathbf{w}) = s(P_d(\mathbf{w})) + s(Q_d(\mathbf{w}))$$

where $tc(\mathbf{w})$ is the twice the sum of the colors in the top line of $\mathbf{w}$.

(2) *The weight of $P_d(\mathbf{w})$ is the weight of the lowest line of $\mathbf{w}$. The weight of $Q_d(\mathbf{w})$ is the weight of the middle line of $\mathbf{w}$.*

In the standard case, Garfinkle's domino insertion is determined by insisting that horizontal dominoes bump by rows and vertical dominoes bump by columns. More precisely, let S be a domino tableaux with no value repeated (but still semistandard), and i some number not used in S. We will describe $(S \leftarrow (0, i))$ and $(S \leftarrow (1, i))$ which correspond to the insertion of a horizontal (color 0) and vertical domino (color 1) labelled i respectively.

Let $T_{<i}$ be the subtableaux of S consisting of all dominoes labelled with numbers less than i. Then set $T_{\leq i}$ to be $T_{<i}$ union a horizontal domino in the first row labelled i or a vertical domino in the first column labelled i depending on what we are inserting. Now for $j > i$ we will recursively define $T_{\leq j}$ given $T_{\leq j-1}$. If there is no domino labelled j in T then $T_{\leq j} = T_{\leq j-1}$. Otherwise let γ_j denote the domino labelled j in S and set $\lambda = \mathrm{sh}(T_{\leq j-1})$. We distinguish four cases.

(1) If $\gamma_j \cap \lambda = \emptyset$ then set $T_{\leq j} = T_{\leq j-1} \cup \gamma_j$.
(2) If $\gamma_j \cap \lambda = \gamma_j$ is a horizontal domino in row k then $T_{\leq j}$ is obtained from $T_{\leq j-1}$ by adding a horizontal domino labelled j to row $k + 1$.

(3) If $\gamma_j \cap \lambda = \gamma_j$ is a vertical domino in column k then $T_{\leq j}$ is obtained from $T_{\leq j-1}$ by adding a vertical domino labelled j to column $k+1$.
(4) If $\gamma_j \cap \lambda = (l, m)$ is a single square then $T_{\leq j}$ is obtained from $T_{\leq j-1}$ by adding a domino labelled j so that the total shape of $T_{\leq j}$ is $\lambda \cup (l+1, m+1)$.

The resulting tableaux $T_{<\infty} = (S \leftarrow (c, i))$.

Figure 5 gives an example of domino insertion.

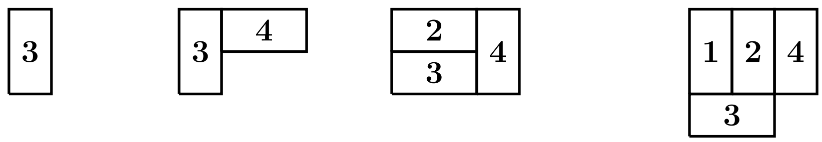

FIGURE 5. The result of the insertion $((((\emptyset \leftarrow (1,3)) \leftarrow (0,4)) \leftarrow (0,2)) \leftarrow (1,1))$.

Shimozono and White's semistandard extension of domino insertion leads automatically to the domino Cauchy formula as observed in [Lam]. In [Lam], we have also described two dual domino insertion algorithms which are bijections between 'dual colored biwords' and pairs of semistandard tableaux of conjugate shape. This proves the dual domino Cauchy formula ($n = 2$ in Theorem 36 here).

It further turns out that Garfinkle's domino insertion has the following domino increasing insertion property. This was first shown by Shimozono and White by connecting domino insertion with mixed insertion. [Lam] gives a different proof using growth diagrams. This domino increasing insertion property can be described by specifying an order $<$ on dominoes as follows (γ_i denotes a domino labelled i)

(1) If γ_i is horizontal and γ_j vertical then $\gamma_i > \gamma_j$.
(2) If γ_i and γ_j are both horizontal then $\gamma_i > \gamma_j$ if and only if $i > j$.
(3) If γ_i and γ_j are both vertical then $\gamma_i > \gamma_j$ if and only if $i < j$.

Under this order, Garfinkle's domino insertion has a ribbon increasing insertion property, as described in Section 3.1.2:

Lemma 49. *Let T be a domino tableaux without the labels i and j. Set $T' = (T \leftarrow \gamma_i)$ and $T'' = (T' \leftarrow \gamma_j)$ for some dominoes γ_i and γ_j. Then $sh(T'/T)$ lies to the left of $sh(T''/T')$ if and only if $\gamma_i < \gamma_j$.*

Similarly, the dual domino insertion has a property which is dual to this. This increasing property is retained when the bijection is extended to the semistandard case which we shall now describe in brief (see [SW, Lam] for details). Let T be a semistandard domino tableaux and (c, j) a domino we want to insert, where j is a value possibly occurring in T. For each value i, all the dominoes labelled i in T can be ordered from left to right and labelled $i_1, i_2, \ldots$, where $i - 1 < i_1 < i_2 < \cdots < i+1$ – all i_a behave like an 'i' when compared to any other value. Then this insertion can be simulated by treating the new label j as being larger than or smaller than all other values j_a (present in T) depending on whether $c = 0$ or $c = 1$. We may then perform insertion as in the standard case. Afterwards we rename all the i_a to

i to obtain $(T \leftarrow (c, j))$. In particular, one may check that the increasing insertion property is compatible with the semistandard insertion.

Immediately we obtain

Proposition 50. *Semistandard domino insertion gives a combinatorial proof of the Pieri rule (Theorem 29) for $n = 2$. Dual semistandard domino insertion gives a combinatorial proof of the dual Pieri rule for $n = 2$.*

3.1.4. **Shimozono and White's ribbon insertion.** Shimozono and White [SW2] have described a ribbon insertion algorithm for general n. This can be described in a traditional bumping fashion or in terms of Fomin's growth diagrams [Fom1, Fom2].

The ribbon insertion algorithm of [SW2] has the usual weight preserving properties, but also the spin to color property (7) which an earlier ribbon-RSK algorithm of Stanton and White [SW1] did not have. However, the algorithm stops short of being a bijection between colored biwords and pairs of semistandard ribbon tableaux. The algorithm is only described as a bijection π between colored words $\mathbf{w}$ (not biwords) and a pair $(P_r(\mathbf{w}), Q_r(\mathbf{w}))$ where $P_r(\mathbf{w})$ is a semistandard ribbon tableaux and $Q_r(\mathbf{w})$ is a standard ribbon tableaux. In particular the Cauchy identity of Theorem 35 does not immediately follow. The algorithm also does not seem to possess a *ribbon increasing insertion* property. However one can at least salvage the following, which is just the first Pieri rule.

Proposition 51. *Shimozono and White's bijection π gives a combinatorial proof that*

$$(1 + q^2 + \ldots + q^{2(n-1)})h_1\mathcal{G}_\lambda(X; q) = \sum_\mu q^{s(\mu/\lambda)}\mathcal{G}_\mu(X; q)$$

where the sum is over all μ such that μ/λ is a n-ribbon.

Proof. As before we construct a weight preserving bijection between the two sides of the Pieri rule by:

$$(T, (c, j)) \mapsto T' = (T \leftarrow (c, j)).$$

The color c ranges from 0 to $n-1$ and h_1 is just the generating function for the labels j. □

Shimozono and White's ribbon insertion is determined by forcing all ribbons to bump by rows to another ribbon of the same spin (at least in the standard case). It is possible however to insist that all ribbons of a particular spin bump by columns instead. Unfortunately, it appears that none of these algorithms have a ribbon increasing insertion property.

3.2. Murnagham-Nakayama and Pieri

In this (self-contained) section we will study the formal combinatorial relationship between Murnagham-Nakayama and Pieri rules for ribbon tableaux.

In particular we will obtain a direct proof that the usual Murnagham-Nakayama rule and Pieri rules are formally equivalent in a combinatorial fashion. This bypasses the usual method of proof which goes via the Jacobi-Trudi formulae. The only algebraic fact needed is the following lemma:

Lemma 52. *The power sum and elementary symmetric functions satisfy the following equation*
$$ne_n = p_1e_{n-1} - p_2e_{n-2} + \cdots + (-1)^{n-1}p_n.$$
Similarly, we have
$$mh_m = p_{m-1}h_1 + p_{m-2}h_2 + \cdots + p_m$$
for the homogenoeous and power sum symmetric functions.

Proof. See (2.10) in [Mac]. $\square$

Let V be a vector space over $\mathbb{C}(q)$ and v_λ be vectors in V labelled by partitions. Recall the definitions of $\mathcal{X}_k^{\mu/\lambda}(q)$, $\mathcal{K}_{\mu/\lambda,k}(q)$ and $\mathcal{L}_{\mu/\lambda,k}(q)$ from Section 2.1. Suppose $\{P_k\}$ are commuting linear operators satisfying
$$P_kv_\lambda = \sum_\mu \mathcal{X}_k^{\mu/\lambda}(q)v_\mu \quad \text{for all } k$$
then we will say that the Murnagham-Nakayama rule holds.

Suppose $\{H_k\}$ are commuting linear operators on V satisfying
$$H_kv_\lambda = \sum_\mu \mathcal{K}_{\mu/\lambda,k}(q)v_\mu \quad \text{for all } k,$$
then we will say that Pieri formula holds.

Suppose $\{E_k\}$ are commuting linear operators on V satisfying
$$E_kv_\lambda = \sum_\mu \mathcal{L}_{\mu/\lambda,k}(q)v_\mu \quad \text{for all } k,$$
then we will say that dual-Pieri formula holds.

If the skew shapes μ/λ are replaced by λ/μ in the above formulae, we get adjoint versions of these formulae which can be thought of as lowering operators. Thus if a set of commuting linear operators $\left\{P_k^\perp\right\}$ satisfies
$$P_k^\perp v_\lambda = \sum_\mu \mathcal{X}_k^{\lambda/\mu}(q)v_\mu \quad \text{for all } k$$
then we will say the lowering Murnagham-Nakayama rule holds, and similarly for $\left\{E_k^\perp\right\}$ and $\left\{H_k^\perp\right\}$.

Proposition 53. *Fix $n \geq 1$ as usual. Let $\{H_k\}$ and $\{P_k\}$ be commuting sets of linear operators satisfying the relations between h_k and p_k in Λ. Then the ribbon Murnagham-Nakayama rule holds for $\{P_k\}$ if and only if the ribbon Pieri rule holds for $\{H_k\}$.*

Proof. Let us suppose the Murnagham-Nakayama rule holds for $\{P_k\}$. We will proceed by induction on k. Since $H_1 = P_1$ the starting condition is clear. Now suppose the proposition has been shown up to $k-1$. By Lemma 52, kH_k acts on v_λ in the same way that $H_{k-1}P_1 + H_{k-2}P_2 + \cdots + P_k$ does.

The action of the latter on v_λ gives some linear combination of v_μ where μ/λ is a union (S,T) of a ribbon border strip S and a horizontal ribbon strip T (where T is a horizontal ribbon strip of the shape $\mu/(\lambda \cup S)$). Denote by $S_1, \ldots, S_a$ the

distinguished decomposition of S into horizontal ribbon strips. Fixing μ we now consider the set $\mathcal{S}$ of such ordered pairs (S, T) where S has size between 1 and k, while T has size from $k-1$ to 0.

We now place the (S, T) into equivalence classes $\mathcal{P}(S, T)$. The equivalence relation is given by taking the transitive closure of the relation

$$(S, T) \sim (S - S_a, T \cup S_a) \tag{8}$$

for every pair such that $T \cup S_a$ is a horizontal strip. This relation is ill-defined when $a = 1$, that is when S is actually a single connected horizontal ribbon strip (in which case $S - S_1$ is empty and $(S - S_1, T \cup S_1)$ does not belong to $\mathcal{S}$), which we shall ignore for the moment.

Let us consider any other equivalence class $\mathcal{P} = \mathcal{P}(S, T)$. We claim that it contains a unique element (S', T') (where $S' = \{S'_1, \ldots, S'_a\}$) such that $T' \cup S'_a$ is not a horizontal ribbon strip. This is due to the definition of a border ribbon strip which ensures that the right hand side of (8) always has this property. Thus the graph of the relations (8) is star-shaped, proving our claim. Now let C be a component of T' such that $C \cup S'_a$ is not a horizontal ribbon strip. Then there is a unique sub-horizontal ribbon strip C' of C which can be added to S' to form a ribbon strip. This C' may be described as follows. Order the ribbons of C from left to right $c_1, c_2, \ldots, c_l$. Find the smallest i such that c_i touches the bottom of S'_a and we set $C' = \{c_1, c_2, \ldots, c_i\}$. We call C a critical component and C' the nice part of C.

Then the equivalence class $\mathcal{P}$ is exactly (S', T') together with the pairs (S, T) such that $S = \{S_1, \ldots, S_{a+1}\}$ where $S_i = S'_i$ for $1 \leq i \leq a$, and S_{a+1} is the union of the nice parts of some (arbitrary) subset of the set critical components of T'. It is immediate from the construction that (S, T) will be a valid pair in $\mathcal{S}$. We observe that the contribution of $\mathcal{P}$

$$\sum_{(S,T) \in \mathcal{P}} (-1)^{h(S)} q^{s(S \cup T)}$$

to the coefficient of v_μ is exactly 0, since the the tiling and hence the spin of the contribution is fixed and the definition of height is exactly so that the signs sum up to 0 (this corresponds to the identity $(1-1)^c = 0$).

It remains to consider the elements (S, T) where S is a connected horizontal ribbon strip such that $S \cup T$ is also a horizontal ribbon strip. Since S is connected we can recover it from $S \cup T$ by specifying its rightmost ribbon. Thus such pairs occur exactly k times for each horizontal ribbon strip of shape μ/λ, and hence the Pieri rule is satisfied for the operator H_k.

The converse clearly follows from the same argument. □

Theorem 54. *Let $\{H_i\}$, $\{E_i\}$ and $\{P_i\}$ be commuting operators on a vector space V over $\mathbb{C}(q)$ satisfy the relations of h_i, e_i and p_i in Λ. Let v_λ be a set of vectors in V indexed by partitions. Suppose that one of the Pieri, dual-Pieri and Murnagham-Nakayama holds, then all three holds. The same is true for the lowering operators satisfying the same relation.*

Proof. That the Murnagham-Nakayama rule and Pieri rules are equivalent is just Proposition 53. One way to see that the Pieri rules and dual-Pieri rules are equivalent is to argue as before and use the relation

$$h_m - h_{m-1}e_1 + \cdots + (-1)^m e_m = 0$$

which is easily deduced from the generating functions $H(t) = \sum_m h_m t^m$ and $E(t) = \sum_m e_m t^m$. However, a short cut is to use Proposition 5. We see that both the Pieri and dual-Pieri formulae hold in $\mathbf{F}$ for operators satisfying the relations of h_i and e_i. But the vectors $|\lambda\rangle$ are linearly independent in $\mathbf{F}$ so this formally implies (by linearity) that the same is true for any set of vectors v_λ in a vector space V over $\mathbb{C}(q)$.

The argument is identical for lowering operators. □

Note that the condition on a horizontal ribbon strip to be connected can be described in terms of the n-quotient as follows. Let T be a ribbon tableaux with n-quotient $\left\{T^{(0)}, \ldots, T^{(n-1)}\right\}$. Let $\{(d_i, p_i)\}$ be the set of diagonals which are nonempty in the n-quotient of the horizontal ribbon strip R. Thus diagonal $diag_{d_i}$ of $T^{(p_i)}$ contains a square corresponding to some ribbon in the horizontal ribbon strip R. Then the horizontal ribbon strip R is connected if and only if the set of integers $\{d_i\}$ is an interval (connected) in $\mathbb{Z}$. Thus border ribbon strips may be characterised in terms of the n-quotient.

References

[BV] D. Barbasch and D. Vogan, Primitive ideals and orbital integrals on complex classical groups, *Math. Ann.*, **259** (1982), 153-199.

[CL] C. Carré and B. Leclerc, Splitting the square of a Schur function into its symmetric and antisymmetric parts, *J. Alg. Combin.*, **4** (1995), 201-231.

[D1] V.V. Deodhar, On some geometric aspects of Bruhat orderings II. The parabolic analogue of Kazhdan-Lusztig polynomials, *J. Algebra*, **111** (1987), 483-506.

[D2] V.V. Deodhar, Duality in parabolic set up for questions in Kazhdan-Lusztig theory, *J. Algebra*, **142** (1991), 201-209.

[Fom1] S. Fomin, Duality of graded graphs, *J. Algebraic Combin.*, **3** (1994), 357-404.

[Fom2] S. Fomin, Schensted algorithms for dual graded graphs, *J. Algebraic Combin.*, **4** (1995), 5-45.

[Gar] D. Garfinkle, On the classification of primitive ideals for complex classical Lie algebras II, *Compositio Math.*, **81** (1992) 307-336.

[GH] A.M. Garsia and M. Haiman, A remarkable q, t-Catalan sequence and q-Lagrange inversion, *J. Algebraic Combin.*, **5** (1996), no.3, 191-244.

[HHLRU] J. Haglund, M. Haiman, N. Loehr, J.B. Remmel, A. Ulyanov, A combinatorial formula for the character of the diagonal coinvariants, preprint, 2003, math.CO/0310424.

[Hay] T. Hayashi, Q-analogues of Clifford and Weyl algebras – Spinor and oscillator representations of quantum enveloping algebras, *Commun. Math. Phys.*, **127** (1990), 129-144.

[Jin] N.H. Jing, Vertex operators and Hall-Littlewood symmetric functions. *Adv. Math.*, **87** (1991), no. 2, 226–248.

[Kas] M. Kashiwara, On crystal bases of the q-analogue of universal enveloping algebras, *Duke Math. J.*, **63** (1991), 465-516.

[Kas1] M. Kashiwara, Global crystal bases of quantum groups, *Duke Math. J.*, **69** (1993), 455-485.

[KMS] M. Kashiwara, T. Miwa and E. Stern, Decomposition of q-deformed Fock spaces, *Slecta Math.*, **1** (1996) 787.

[KT] M. Kashiwara and T. Tanisaki, Parabolic Kazhdan-Lusztig polynomials and Schubert varieties, *J. Algebra*, **249** (2002), no.2, 306-325.

[KS] A.N. Kirillov and M. Shimozono, A generalization of the Kostka-Foulkes polynomials, *J. Algebraic Combin.*, **15** (2002), no. 1, 27-69.

[KLLT] A.N. Kirillov, A. Lascoux, B. Leclerc, and J.-Y. Thibon, Séries génératrices pour les tableaux de dominos, *C. R. Acad. Sci. Paris*, Série I **318** (1994), 395-400.

[Lam] T. Lam, Growth diagrams, Domino insertion and Sign-imbalance, preprint, 2003, math.CO/0308265.

[LLT1] A. Lascoux, B. Leclerc, and J.-Y. Thibon, Hecke algebras at roots of unity and crystal bases of quantum affine algebras, *Commun. Math. Phys.*, **181** (1996), 205-263.

[LLT] A. Lascoux, B. Leclerc, and J.-Y. Thibon, Ribbon tableaux, Hall-Littlewood symmetric functions, quantum affine algebras, and unipotent varieties, *J. Math. Phys.*, **38**(3) (1997), 1041-1068.

[Lec] B. Leclerc Symmetric functions and the Fock space representation of $U_q(\widehat{\mathfrak{sl}}_n)$, *Lectures given at the Isaac Newton Institute*, (2001).

[LT] B. Leclerc and J.-Y. Thibon, Littlewood-Richardson coefficients and Kazhdan-Lusztig polynomials; *Combinatorial Methods in Representation Theory*, Advanced Studies in Pure Mathematics 28, (2000), 155-220.

[LT1] B. Leclerc and J.-Y. Thibon, Canonical bases of q-deformed Fock spaces, *Int. Math. Res. Notices*, **9** (1996), 447-456.

[vL] M. vanLeeuwen, The Robinson-Schensted and Schutzenberger algorithms, an elementary approach, The Foata Festscrhift, *Electron. J. Combin.*, **3(2)** (1996), Research Paper 15.

[Lit] D.E. Littlewood, Modular representations of symmetric groups, *Proc. Royal Soc. London Ser. A*, **209** (1951), 333-353.

[Mac] I. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press, 1995.

[Mat] A. Mathas, Iwahori-Hecke algebras and Schur algebras of the symmetric group, *University lecture series*, **15**, AMS, 1999.

[MM] K. Misra and T. Miwa, Crystal base for the basic representation of $U_q(\widehat{\mathfrak{sl}}(n))$, *Commun. Math. Phys.*, **134** (1990), 79-88.

[Pie] M. Pieri, Sul problema degli spazi secanti. Nota 1^a, *Rend. Ist. Lombardo*,(2) **26** (1893), 534-546.

[Sch] A. Schilling, q-Supernomial coefficients: From riggings to ribbons, *MathPhys Odyssey 2001, M. Kashiwara and T. Miwa (eds.)*, Birkhaeuser Boston, Cambridge, MA, 2002, 437-454.

[SchW] A. Schilling and S.O. Warnaar, Inhomogeneous lattice paths, generalized Kostka polynomials and A_{n-1} supernomials, *Comm. Math. Phys.*, **202** (1999), no. 2, 359–401.

[SSW] A. Schilling, M. Shimozono and D.E. White, Branching formula for q-Littlewood-Richardson coefficients, *Advances in Applied Mathematics*, **30** (2003), 258-272.

[Shi] M. Shimozono, Affine type A crystal structure on tensor products of rectangles, Demazure characters, and nilpotent varieties. *J. Algebraic Combin.*, **15** (2002), no.2, 151-187.

[SW3] M. Shimozono and J. Weyman, Characters of modules supported in the closure of a nilpotent conjugacy class, *European J. Combin.*, **21** (2000), no. 2, 257-288.

[SW] M. Shimozono and D.E. White, A color-to-spin domino Schensted algorithm, *Electron. J. Combinatorics*, **8** (2001).

[SW2] M. Shimozono and D.E. White, Color-to-spin ribbon Schensted algorithms, *Discrete Math.*, **246** (2002), 295-316.

[SZ] M. Shimozono and M. Zabrocki, Hall-Littlewood vertex operators and generalized Kostka polynomials, *Adv. Math.*, **158** (2001), no. 1, 66-85.

[EC2] R. Stanley, *Enumerative Combinatorics, Vol 2*, Cambridge, 1999.
[Sta] R. Stanley, Some Remarks on Sign-Balanced and Maj-Balanced Posets, preprint, 2002; math.CO/0211113.
[SW1] D. Stanton and D. White, A Schensted algorithm for rim-hook tableaux, *J. Combin. Theory Ser. A.*, **40** (1985), 211-247.
[VV] M. Varagnolo, E. Vasserot, On the decomposition matrices of the quantized Schur algebra, Duke Math. J., **100** (1999), 267-297.
[Whi] D. White, Sign-balanced posets, *J. Combin. Theory Ser. A*, **95** (2001), no. 1, 1-38.

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